

Limit theorems for conditioned non-generic Galton-Watson trees

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Abstract

We study a particular type of subcritical Galton-Watson trees, which are called non-generic trees in the physics community. In contrast with the critical or supercritical case, it is known that condensation appears in certain large conditioned non-generic trees, meaning that with high probability there exists a unique vertex with macroscopic degree comparable to the total size of the tree. We investigate this phenomenon by studying scaling limits of such trees. Using recent results concerning subexponential distributions, we study the convergence of three functions coding these trees (the Lukasiewicz path, the contour function and the height function) and show that the situation is completely different from the critical case. In particular, the height of such trees grows logarithmically in their size. We also study fluctuations around the condensation vertex.

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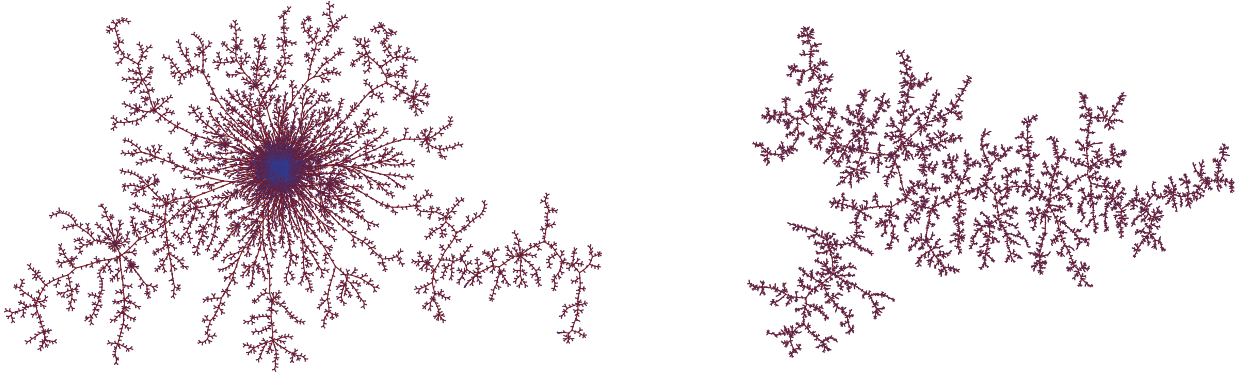


Figure 1: The first figure shows a large non-generic Galton-Watson tree. The second figure shows a large critical Galton-Watson tree with finite variance.

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Introduction

The behavior of large Galton-Watson trees whose offspring distribution $\mu = (\mu_i)_{i \geq 0}$ is *critical* (meaning that the mean of μ is 1) and has *finite variance* has drawn a lot of attention. If \mathbf{t}_n is a Galton-Watson tree with offspring distribution μ (in short a GW_μ tree) conditioned on having total size n , Kesten [23] proved that \mathbf{t}_n converges locally in distribution as $n \rightarrow \infty$ to the so-called critical Galton-Watson tree conditioned to survive. Aldous [1] studied the scaled asymptotic behavior of \mathbf{t}_n by showing that the appropriately rescaled contour function of \mathbf{t}_n converges to the Brownian excursion.

These results have been extended in different directions. The “finite second moment” condition on μ has been relaxed by Duquesne [11], who showed that when μ belongs to the domain of attraction of a stable law of index $\theta \in (1, 2]$, the appropriately rescaled contour function of \mathbf{t}_n converges toward the normalized excursion of the θ -stable height process, which codes the so-called θ -stable tree (see also [25]). In a different direction, several authors have considered trees conditioned by other quantities than the total size, for example by the height [24, 29] or the number of leaves [32, 26].

Non critical Galton-Watson trees. Kennedy [22] noticed that, under certain conditions, the study of non-critical offspring distributions reduces to the study of critical ones. More precisely, if $\lambda > 0$ is a fixed parameter such that $Z_\lambda = \sum_{i \geq 0} \mu_i \lambda^i < \infty$, set $\mu_i^{(\lambda)} = \mu_i \lambda^i / Z_\lambda$ for $i \geq 0$. Then a GW_μ tree conditioned on having total size n has the same distribution as a $\text{GW}_{\mu^{(\lambda)}}$ tree conditioned on having total size n . Thus, if one can find $\lambda > 0$ such that both $Z_\lambda < \infty$ and $\mu^{(\lambda)}$ is critical, then studying a conditioned non-critical Galton-Watson tree boils down to studying a critical one. This explains why the critical case has been extensively studied in the literature.

Let μ be a probability distribution such that $\mu_0 > 0$ and $\mu_k > 0$ for some $k \geq 2$. We are interested in the case where there exist *no* $\lambda > 0$ such that both $Z_\lambda < \infty$ and $\mu^{(\lambda)}$ is critical (see [18, Section 8] for a characterization of such probability distributions). An important example is when μ is subcritical (i.e. of mean strictly less than 1) and $\mu_i \sim c/i^\beta$ as $i \rightarrow \infty$ for a fixed parameter $\beta > 2$. The study of such GW_μ trees conditioned on having a large fixed size was initiated only recently by Jonsson & Stefánsson [21] who called such trees *non-generic* trees. They studied the above-mentioned case where $\mu_i \sim c/i^\beta$ as $i \rightarrow \infty$, with $\beta > 2$, and showed that if \mathbf{t}_n is a GW_μ tree conditioned on having total size n , then with probability tending to 1 as $n \rightarrow \infty$, there exists a unique vertex of \mathbf{t}_n with maximal degree, which is asymptotic to $(1 - \mathbf{m})n$ where $\mathbf{m} < 1$ is the mean of μ . This phenomenon is called *condensation* and appears in a variety of statistical mechanics models such as the Bose-Einstein condensation for bosons, the zero-range process [20, 13] or the Backgammon model [4] (see Fig. 1).

Jonsson and Stefánsson [21] have also constructed an infinite random tree $\widehat{\mathcal{T}}$ (with a unique vertex of infinite degree) such that \mathbf{t}_n converges locally in distribution toward $\widehat{\mathcal{T}}$ (meaning roughly that the degree of every vertex of \mathbf{t}_n converges toward the degree of the corresponding vertex of $\widehat{\mathcal{T}}$). See Proposition 2.7 below for the description of $\widehat{\mathcal{T}}$. In [18], Janson has extended this result to simply generated trees and has in particular given a very precise description of local properties of Galton-Watson trees conditioned on their size.

In this work, we are interested in the existence of scaling limits for the random trees \mathbf{t}_n . When scaling limits exist, one often gets information concerning the global structure of the tree.

Notation and assumptions. Throughout this work $\theta > 1$ will be a fixed parameter. We say that a probability distribution $(\mu_j)_{j \geq 0}$ on the nonnegative integers satisfies Assumption (H_θ) if the following two conditions hold:

- (i) μ is subcritical, meaning that $0 < \sum_{j=0}^{\infty} j\mu_j < 1$.
- (ii) There exists a function $\mathcal{L} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\mathcal{L}(x) > 0$ for x large enough and $\lim_{x \rightarrow \infty} \mathcal{L(tx)}/\mathcal{L}(x) = 1$ for all $t > 0$ (such a function is called slowly varying) and $\mu_n = \mathcal{L}(n)/n^{1+\theta}$ for every $n \geq 1$.

Let $\zeta(\tau)$ be the total progeny or size of a tree τ . Condition (ii) implies that $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$ for sufficiently large n . Note that (ii) is more general than the analogous assumption in [21, 18], where only the case $\mathcal{L}(x) \rightarrow c$ as $x \rightarrow \infty$ was studied in detail. Throughout this text, $\theta > 1$ is a fixed parameter and μ is a probability distribution on \mathbb{N} satisfying the Assumption (H_θ) . In addition, for every $n \geq 1$ such that $\mathbb{P}_\mu[\zeta(\tau) = n] > 0$, \mathbf{t}_n is a GW_μ tree conditioned on having n vertices (note that \mathbf{t}_n is well defined for n sufficiently large). The mean of μ will be denoted by \mathbf{m} and we set $\gamma = 1 - \mathbf{m}$.

We are now ready to state our main results which concern different aspect of non-generic trees. We are first interested in the condensation phenomenon and derive properties of the maximal degree. We then find the location of the vertex of maximal degree. Finally we investigate the global behavior of non-generic trees by studying their height.

Condensation. If τ is a (finite) tree, we denote by $\Delta(\tau)$ the maximal out-degree of a vertex of τ (the out-degree of a vertex is by definition its number of children). If τ is a finite tree, let $u_\star(\tau)$ be the smallest vertex (in the lexicographical order, see Definition 1.3 below) of τ with maximal out-degree. The following result states that, with probability tending to 1 as $n \rightarrow \infty$, there exists a vertex of \mathbf{t}_n with out-degree roughly γn and that the deviations around this typical value are of order roughly $n^{1/(2 \wedge \theta)}$, and also that the out-degrees of all the other vertices of \mathbf{t}_n are roughly of order $n^{1/(2 \wedge \theta)}$. In particular the vertex with maximal out-degree is unique with probability tending to 1 as $n \rightarrow \infty$.

Theorem 1. *There exists a slowly varying function L such that if $B_n = L(n)n^{1/(2 \wedge \theta)}$, the following assertions hold:*

- (i) We have $\frac{\Delta(\mathbf{t}_n)}{\gamma n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1$.
- (ii) With probability tending to 1 as $n \rightarrow \infty$, the out-degrees of all the vertices of \mathbf{t}_n , except $u_\star(\mathbf{t}_n)$, are $O(B_n)$.
- (iii) Let $(Y_t)_{t \geq 0}$ be a spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda Y_t)] = \exp(t\lambda^{2 \wedge \theta})$. Then

$$\frac{\Delta(\mathbf{t}_n) - \gamma n}{B_n} \xrightarrow[n \rightarrow \infty]{(d)} -Y_1.$$

When μ has finite variance $\sigma^2 \in (0, \infty)$, one may take $B_n = \sigma\sqrt{n/2}$. Theorem 1 has already been proved when $\mu_n \sim c/n^{1+\theta}$ as $n \rightarrow \infty$ (that is when $\mathcal{L} = c + o(1)$, in which case one may

choose L to be a constant function) by Jonsson & Stefánsson [21] for (i) and Janson [18] for (ii) and (iii). However, our techniques are different and are based on a coding of \mathbf{t}_n by a conditioned random walk combined with recent results of Armendáriz & Loulakis [2] concerning random walks whose jump distribution is subexponential. The main advantage of this approach is that it enables us to obtain new results concerning the structure of \mathbf{t}_n .

Localization of the vertex of maximal degree. We are also interested in the location of the vertex of maximal degree $u_*(\mathbf{t}_n)$. Before stating our results, we need to introduce some notation. If τ is a tree, let $U(\tau)$ be the index in the lexicographical order of the first vertex of τ with maximal out-degree (when the vertices of τ are ordered starting from index 0). Note that the number of children of $u_*(\tau)$ is $\Delta(\tau)$. Denote the generation of $u_*(\tau)$ by $|u_*(\tau)|$. Finally, if $u \in \tau$, the pointed tree τ cut at u is by definition the subtree of τ obtained by removing all the descendants of u , pointed at u (meaning that the vertex u is distinguished).

Theorem 2. *The following three convergences hold:*

$$(i) \text{ For } i \geq 0, \mathbb{P}[U(\mathbf{t}_n) = i] \xrightarrow{n \rightarrow \infty} \gamma \cdot \mathbb{P}_\mu[\zeta(\tau) \geq i + 1].$$

$$(ii) \text{ For } i \geq 0, \mathbb{P}[|u_*(\mathbf{t}_n)| = i] \xrightarrow{n \rightarrow \infty} (1 - \mathbf{m})\mathbf{m}^i.$$

(iii) *As $n \rightarrow \infty$, the tree \mathbf{t}_n cut at $u_*(\mathbf{t}_n)$ converges locally in distribution toward the tree $\hat{\mathcal{T}}$ cut at its vertex of infinite degree.*

See Proposition 2.7 below for the description of $\hat{\mathcal{T}}$. Using completely different techniques, a result similar to assertion (i) has been proved by Durrett [12, Theorem 3.2] in the context of random walks when \mathbf{t}_n is a GW_μ tree conditioned on having *at least* n vertices and in addition μ has finite variance. However, the so-called local conditioning by having a fixed number of vertices is often much more difficult to analyze (see e.g. [11, 29]). Note that $\sum_{i \geq 1} \mathbb{P}_\mu[\zeta(\tau) \geq i] = \mathbb{E}_\mu[\zeta(\tau)] = 1/\gamma$, so that the limit in (i) is a probability distribution. The proof of (i) combines the coding of \mathbf{t}_n by a conditioned random walk with the use of local limit theorems. The proof of the other two assertions uses (i) together with the local convergence of \mathbf{t}_n toward the infinite random tree $\hat{\mathcal{T}}$, which has been obtained by Jonsson & Stefánsson [21] in a particular case and then generalized by Janson [18], and was already mentioned above.

A key consequence of Theorem 2 is that, informally, the tree \mathbf{t}_n looks like a finite spine of geometric length decorated with independent GW_μ trees, and on top of which are grafted $\Delta(\mathbf{t}_n)$ independent GW_μ trees.

We are also interested in the size of the subtrees grafted on $u_*(\mathbf{t}_n)$. If τ is a tree, for $1 \leq j \leq \Delta(\tau)$, let $\zeta_j(\tau)$ be the number of descendants of the j -th child of $u_*(\tau)$ and set $Z_j(\tau) = \zeta_1(\tau) + \zeta_2(\tau) + \cdots + \zeta_j(\tau)$. If I is an interval, we let $\mathbb{D}(I, \mathbb{R})$ denote the space of all right-continuous with left limits (càdlàg) functions $I \rightarrow \mathbb{R}$, endowed with the Skorokhod J_1 -topology (see [5, chap. 3] and [16, chap. VI] for background concerning the Skorokhod topology). If $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x . Recall that $(Y_t)_{t \geq 0}$ is the spectrally positive Lévy process with Laplace exponent $\mathbb{E}[\exp(-\lambda Y_t)] = \exp(t\lambda^{2 \wedge \theta})$. Recall the sequence $(B_n)_{n \geq 1}$ from Theorem 1.

Theorem 3. *The following convergence holds in distribution in $\mathbb{D}([0, 1], \mathbb{R})$:*

$$\left(\frac{Z_{\lfloor \Delta(\mathbf{t}_n)t \rfloor}(\mathbf{t}_n) - \Delta(\mathbf{t}_n)t/\gamma}{B_n}, 0 \leq t \leq 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} \left(\frac{1}{\gamma} Y_t, 0 \leq t \leq 1 \right).$$

Note that in the case when μ has finite variance, we have $\theta \geq 2$ and Y is just a constant times standard Brownian motion. Let us mention that Theorem 3 is used in [19] to study scaling limits of random planar maps with a unique large face (see [19, Proposition 3.1]) and is also used in [8] to study the shape of large supercritical percolation clusters on random triangulations.

Corollary 1. *If $\theta \geq 2$, $\max_{1 \leq i \leq \Delta(\mathbf{t}_n)} \zeta_i(\mathbf{t}_n)/B_n$ converges in probability toward 0 as $n \rightarrow \infty$. If $\theta < 2$, for every $u > 0$ we have:*

$$\mathbb{P} \left[\frac{1}{B_n} \max_{1 \leq i \leq \Delta(\mathbf{t}_n)} \zeta_i(\mathbf{t}_n) \leq u \right] \xrightarrow[n \rightarrow \infty]{} \exp \left(\frac{1}{\gamma^\theta \Gamma(1 - \theta)} u^{-\theta} \right),$$

where Γ is Euler's Gamma function.

Height of non-generic trees. One of the main contributions of this work is to understand the growth of the height $\mathcal{H}(\mathbf{t}_n)$ of \mathbf{t}_n , which is by definition the maximal generation in \mathbf{t}_n . We establish the key fact that $\mathcal{H}(\mathbf{t}_n)$ grows logarithmically in n :

Theorem 4. *For every sequence $(\lambda_n)_{n \geq 1}$ of positive real numbers tending to infinity:*

$$\mathbb{P} \left[\left| \mathcal{H}(\mathbf{t}_n) - \frac{\ln(n)}{\ln(1/\mathbf{m})} \right| \leq \lambda_n \right] \xrightarrow[n \rightarrow \infty]{} 1.$$

Note that the situation is completely different from the critical case, where $\mathcal{H}(\mathbf{t}_n)$ grows like a power of n . Theorem 4 implies that $\mathcal{H}(\mathbf{t}_n)/\ln(n) \rightarrow \ln(1/\mathbf{m})$ in probability as $n \rightarrow \infty$, thus partially answering Problem 20.7 in [18]. Theorem 4 can be intuitively explained by the fact that the height of \mathbf{t}_n should be close to the maximum of the height of γn independent subcritical GW_μ trees, which is indeed of order $\ln(n)$.

Since \mathbf{t}_n grows roughly as $\ln(n)$ as $n \rightarrow \infty$, it is natural to wonder if one could hope to obtain a scaling limit after rescaling the distances in \mathbf{t}_n by $\ln(n)$. We show that the answer is negative and that we cannot hope to obtain a nontrivial scaling limit for \mathbf{t}_n for the Gromov-Hausdorff topology, in sharp contrast with the critical case (see [11]). This partially answers a question of Janson [18, Problem 20.11].

Theorem 5. *The sequence $(\ln(n)^{-1} \cdot \mathbf{t}_n)_{n \geq 1}$ is not tight for the Gromov-Hausdorff topology, where $\ln(n)^{-1} \cdot \mathbf{t}_n$ stands for the metric space obtained from \mathbf{t}_n by multiplying all distances by $\ln(n)^{-1}$.*

The Gromov-Hausdorff topology is the topology on compact metric spaces (up to isometries) defined by the Gromov-Hausdorff distance, and is often used in the study of scaling limits of different classes of random graphs (see [7, Chapter 7] for background concerning the Gromov-Hausdorff topology).

However, we establish the convergence of the finite-dimensional marginal distributions of the height function coding \mathbf{t}_n . If τ is a tree, for $0 \leq i \leq \zeta(\tau) - 1$, denote by $H_i(\tau)$ the generation of the i -th vertex of τ in the lexicographical order.

Theorem 6. *Let $k \geq 1$ be an integer and fix $0 < t_1 < \dots < t_k < 1$. Then*

$$(H_{\lfloor nt_1 \rfloor}(\mathbf{t}_n), H_{\lfloor nt_2 \rfloor}(\mathbf{t}_n), \dots, H_{\lfloor nt_k \rfloor}(\mathbf{t}_n)) \xrightarrow[n \rightarrow \infty]{(d)} (1 + \mathbf{e}_0 + \mathbf{e}_1, 1 + \mathbf{e}_0 + \mathbf{e}_2, \dots, 1 + \mathbf{e}_0 + \mathbf{e}_k),$$

where $(\mathbf{e}_i)_{i \geq 0}$ is a sequence of i.i.d. geometric random variables of parameter $1 - \mathbf{m}$ (that is $\mathbb{P}[\mathbf{e}_0 = i] = (1 - \mathbf{m})\mathbf{m}^i$ for $i \geq 0$).

Informally, the random variable \mathbf{e}_0 describes the length of the spine, and the random variables $(\mathbf{e}_1, \dots, \mathbf{e}_k)$ describe the height of vertices chosen in a forest of independent subcritical GW_μ trees. Note that these finite-dimensional marginal distributions converge without scaling, even though the height of \mathbf{t}_n is of order $\ln(n)$.

This text is organized as follows. We first recall the definition and basic properties of Galton-Watson trees. In Section 2, we establish limit theorems for large conditioned non-generic Galton-Watson trees. We conclude by giving possible extensions and formulating some open problems.

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1 Galton-Watson trees

1.1 Basic definitions

We briefly recall the formalism of plane trees (also known in the literature as rooted ordered trees) which can be found in [28] for example.

Definition 1.1. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of all nonnegative integers and let \mathbb{N}^* be the set of all positive integers. Let also U be the set of all labels defined by:

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

where by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. An element of U is a sequence $u = u_1 \cdots u_k$ of positive integers and we set $|u| = k$, which represents the “generation” of u . If $u = u_1 \cdots u_i$ and $v = v_1 \cdots v_j$ belong to U , we write $uv = u_1 \cdots u_i v_1 \cdots v_j$ for the concatenation of u and v . In particular, we have $u\emptyset = \emptyset u = u$. Finally, a *plane tree* τ is a finite or infinite subset of U such that:

- (i) $\emptyset \in \tau$,
- (ii) if $v \in \tau$ and $v = ui$ for some $i \in \mathbb{N}^*$, then $u \in \tau$,
- (iii) for every $u \in \tau$, there exists $k_u(\tau) \in \{0, 1, 2, \dots\} \cup \{\infty\}$ (the number of children of u) such that, for every $j \in \mathbb{N}^*$, $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

Note that in contrast with [27, 28] we allow the possibility $k_u(\tau) = \infty$ in (iii). In the following, by *tree* we will always mean plane tree, and we denote the set of all trees by \mathbb{T} . We will often view each vertex of a tree τ as an individual of a population whose τ is the genealogical tree. The total progeny or size of τ will be denoted by $\zeta(\tau) = \text{Card}(\tau)$. If τ is a tree and $u \in \tau$, we define the shift of τ at u by $T_u\tau = \{v \in U; uv \in \tau\}$, which is itself a tree.

Definition 1.2. Let ρ be a probability measure on \mathbb{N} . The law of the Galton-Watson tree with offspring distribution ρ is the unique probability measure \mathbb{P}_ρ on \mathbb{T} such that:

- (i) $\mathbb{P}_\rho[k_\emptyset = j] = \rho(j)$ for $j \geq 0$,
- (ii) for every $j \geq 1$ with $\rho(j) > 0$, conditionally on $\{k_\emptyset = j\}$, the subtrees $T_1\tau, \dots, T_j\tau$ are independent and identically distributed with distribution \mathbb{P}_ρ .

A random tree whose distribution is \mathbb{P}_ρ will be called a Galton-Watson tree with offspring distribution ρ , or in short a GW_ρ tree.

In the sequel, for every integer $j \geq 1$, $\mathbb{P}_{\rho,j}$ will denote the probability measure on \mathbb{T}^j which is the distribution of j independent GW_ρ trees. The canonical element of \mathbb{T}^j is denoted by \mathbf{f} . For $\mathbf{f} = (\tau_1, \dots, \tau_j) \in \mathbb{T}^j$, let $\zeta(\mathbf{f}) = \zeta(\tau_1) + \dots + \zeta(\tau_j)$ be the total progeny of \mathbf{f} .

1.2 Coding Galton-Watson trees

We now explain how trees can be coded by three different functions. These codings are important in the understanding of large Galton-Watson trees.

Definition 1.3. We write $u < v$ for the lexicographical order on the labels U (for example $\emptyset < 1 < 21 < 22$). Let τ be a finite tree and order the individuals of τ in lexicographical order: $\emptyset = u(0) < u(1) < \dots < u(\zeta(\tau) - 1)$. The height process $H(\tau) = (H_n(\tau), 0 \leq n < \zeta(\tau))$ is defined, for $0 \leq n < \zeta(\tau)$, by:

$$H_n(\tau) = |u(n)|.$$

We set $H_{\zeta(\tau)}(\tau) = 0$ for technical reasons. The height $\mathcal{H}(\tau)$ of τ is by definition $\max_{0 \leq n < \zeta(\tau)} H_n(\tau)$.



Figure 2: A tree τ with its vertices indexed in lexicographical order and its contour function $(C_u(\tau); 0 \leq u \leq 2(\zeta(\tau) - 1))$. Here, $\zeta(\tau) = 26$.

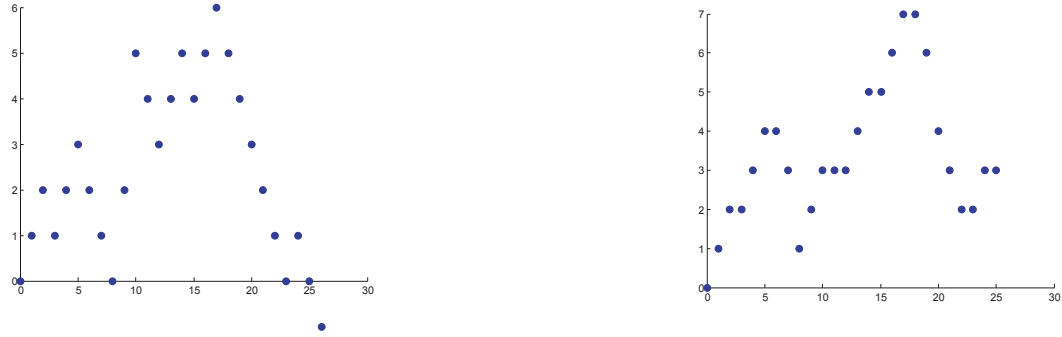


Figure 3: The Lukasiewicz path $W(\tau)$ and the height function $H(\tau)$ of τ .

Consider a particle that starts from the root and visits continuously all the edges of τ at unit speed, assuming that every edge has unit length. When the particle leaves a vertex, it moves toward the first non visited child of this vertex if there is such a child, or returns to the parent of this vertex. Since all the edges are crossed twice, the total time needed to explore the tree is $2(\zeta(\tau) - 1)$. For $0 \leq t \leq 2(\zeta(\tau) - 1)$, $C_\tau(t)$ is defined as the distance to the root of the position of the particle at time t . For technical reasons, we set $C_\tau(t) = 0$ for $t \in [2(\zeta(\tau) - 1), 2\zeta(\tau)]$. The function $C(\tau)$ is called the contour function of the tree τ . See Figure 3 for an example, and [11, Section 2] for a rigorous definition.

Finally, the Lukasiewicz path $W(\tau) = (W_n(\tau), 0 \leq n \leq \zeta(\tau))$ of τ is defined by $W_0(\tau) = 0$ and for $0 \leq n \leq \zeta(\tau) - 1$:

$$W_{n+1}(\tau) = W_n(\tau) + k_{u(n)}(\tau) - 1.$$

Note that necessarily $W_{\zeta(\tau)}(\tau) = -1$ and that $U(\mathbf{t}_n) = \min\{j \geq 0; W_{j+1}(\mathbf{t}_n) - W_j(\mathbf{t}_n) = \Delta(\mathbf{t}_n) - 1\}$, where we recall that $U(\tau)$ is the index in the lexicographical order of the first vertex of \mathbf{t}_n with maximal out-degree.

The following proposition explains the importance of the Lukasiewicz path. Let ρ be a critical or subcritical probability distribution on \mathbb{N} with $\rho(1) < 1$.

Proposition 1.4. *Let $(W_n)_{n \geq 0}$ be a random walk with starting point $W_0 = 0$ and jump distribution $\nu(k) = \rho(k + 1)$ for $k \geq -1$. Set $\zeta = \inf\{n \geq 0; W_n = -1\}$. Then $(W_0, W_1, \dots, W_\zeta)$ has the same distribution as the Lukasiewicz path of a GW_ρ tree. In particular, the total progeny of a GW_ρ tree has the same law as ζ .*

Proof. See [27, Proposition 1.5]. □

1.3 The Vervaat transformation

For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ and $i \in \mathbb{Z}/n\mathbb{Z}$, denote by $\mathbf{x}^{(i)}$ the i -th cyclic shift of \mathbf{x} defined by $x_k^{(i)} = x_{i+k \bmod n}$ for $1 \leq k \leq n$.

Definition 1.5. Let $n \geq 1$ be an integer and let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$. Set $w_j = x_1 + \dots + x_j$ for $1 \leq j \leq n$ and let the integer $i_*(\mathbf{x})$ be defined by $i_*(\mathbf{x}) = \inf\{j \geq 1; w_j = \min_{1 \leq i \leq n} w_i\}$. The Vervaat transform of \mathbf{x} , denoted by $\mathbf{V}(\mathbf{x})$, is defined to be $\mathbf{x}^{(i_*(\mathbf{x}))}$.

The following fact is well known (see e.g. [30, Section 5]):

Proposition 1.6. Let $(W_n, n \geq 0)$ be as in Proposition 1.4 and $X_k = W_k - W_{k-1}$ for $k \geq 1$. Fix an integer $n \geq 1$ such that $\mathbb{P}[W_n = -1] > 0$. The law of $\mathbf{V}(X_1, \dots, X_n)$ under $\mathbb{P}[\cdot | W_n = -1]$ coincides with the law of (X_1, \dots, X_n) under $\mathbb{P}[\cdot | \zeta = n]$.

From Proposition 1.4, it follows that the law of $\mathbf{V}(X_1, \dots, X_n)$ under $\mathbb{P}[\cdot | W_n = -1]$ coincides with the law of $(\mathcal{W}_1(\mathbf{t}_n), \mathcal{W}_2(\mathbf{t}_n) - \mathcal{W}_1(\mathbf{t}_n), \dots, \mathcal{W}_n(\mathbf{t}_n) - \mathcal{W}_{n-1}(\mathbf{t}_n))$ where \mathbf{t}_n is a GW_ρ tree conditioned on having total progeny equal to n .

We now introduce the Vervaat transformation in continuous time.

Definition 1.7. Set $\mathbb{D}_0([0, 1], \mathbb{R}) = \{\omega \in \mathbb{D}([0, 1], \mathbb{R}); \omega(0) = 0\}$. The Vervaat transformation in continuous time, denoted by $\mathcal{V} : \mathbb{D}_0([0, 1], \mathbb{R}) \rightarrow \mathbb{D}([0, 1], \mathbb{R})$, is defined as follows. For $\omega \in \mathbb{D}_0([0, 1], \mathbb{R})$, set $g_1(\omega) = \inf\{t \in [0, 1]; \omega(t-) \wedge \omega(t) = \inf_{[0, 1]} \omega\}$. Then define:

$$\mathcal{V}(\omega)(t) = \begin{cases} \omega(g_1(\omega) + t) - \inf_{[0, 1]} \omega, & \text{if } g_1(\omega) + t \leq 1, \\ \omega(g_1(\omega) + t - 1) + \omega(1) - \inf_{[0, 1]} \omega & \text{if } g_1(\omega) + t \geq 1. \end{cases}$$

By combining the Vervaat transformation with limit theorems under the conditional probability distribution $\mathbb{P}[\cdot | W_n = -1]$ and using Proposition 1.4 we will obtain information about conditioned Galton-Watson trees. The advantage of dealing with $\mathbb{P}[\cdot | W_n = -1]$ is to avoid any positivity constraint.

1.4 Slowly varying functions

Recall that a function $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be slowly varying if $L(x) > 0$ for x large enough and $\lim_{x \rightarrow \infty} L(tx)/L(x) = 1$ for all $t > 0$. Let $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a slowly varying function. Without further notice, we will use the following standard facts:

- (i) The convergence $\lim_{x \rightarrow \infty} L(tx)/L(x) = 1$ holds uniformly for t in a compact subset of $(0, \infty)$.
- (ii) Fix $\epsilon > 0$. There exists a constant $C > 1$ such that $\frac{1}{C}x^{-\epsilon} \leq L(nx)/L(n) \leq Cx^\epsilon$ for every integer n sufficiently large and $x \geq 1$.

These results are immediate consequences of the so-called representation theorem for slowly varying functions (see e.g. [6, Theorem 1.3.1]).

2 Limit theorems for conditioned non-generic Galton-Watson trees

In the sequel, $(W_n; n \geq 0)$ denotes the random walk introduced in Proposition 1.4 with $\rho = \mu$. Note that $\mathbb{E}[W_1] = -\gamma < 0$. Set $X_0 = 0$ and $X_k = W_k - W_{k-1}$ for $k \geq 1$. It will be convenient to work with centered random walks, so we also set $\bar{W}_n = W_n + \gamma n$ and $\bar{X}_n = X_n + \gamma$ for $n \geq 0$ so that $\bar{W}_n = \bar{X}_1 + \dots + \bar{X}_n$. Obviously, $W_n = -1$ if and only if $\bar{W}_n = \gamma n - 1$.

2.1 Invariance principles for conditioned random walks

In this section, our goal is to prove Theorem 1. We first introduce some notation. Denote by $T : \cup_{n \geq 1} \mathbb{R}^n \rightarrow \cup_{n \geq 1} \mathbb{R}^n$ the operator that interchanges the last and the (first) maximal component of a finite sequence of real numbers:

$$T(x_1, \dots, x_n)_k = \begin{cases} \max_{1 \leq i \leq n} x_i & \text{if } k = n \\ x_n & \text{if } x_k > \max_{1 \leq i < k} x_i \text{ and } x_k = \max_{k \leq i \leq n} x_i \\ x_k & \text{otherwise.} \end{cases}$$

Since μ satisfies Assumption (H_θ) , we have $\mathbb{P}[\bar{W}_1 \in (x, x+1)] \sim \mathcal{L}(x)/x^{1+\theta}$ as $x \rightarrow \infty$. Then, by [10, Theorem 9.1], we have:

$$\mathbb{P}[\bar{W}_n \in (x, x+1)] \underset{n \rightarrow \infty}{\sim} n \mathbb{P}[\bar{W}_1 \in (x, x+1)], \quad (1)$$

uniformly in $x \geq \epsilon n$ for every fixed $\epsilon > 0$. In other words, the distribution of \bar{W}_1 is $(0, 1]$ -subexponential, so that we can apply a recent result of Armendáriz & Loulakis [2] concerning conditioned random walks with subexponential jump distribution. In our particular case, this result can be stated as follows:

Proposition 2.1 (Armendáriz & Loulakis, Theorem 1 in [2]). *For $n \geq 1$ and $x > 0$, let $\mu_{n,x}$ be the probability measure on \mathbb{R}^n which is the distribution of $(\bar{X}_1, \dots, \bar{X}_n)$ under the conditional probability distribution $\mathbb{P}[\cdot | \bar{W}_n \in (x, x+1)]$.*

Then for every $\epsilon > 0$, we have:

$$\lim_{n \rightarrow \infty} \sup_{x \geq \epsilon n} \sup_{A \in \mathfrak{B}(\mathbb{R}^{n-1})} \left| \mu_{n,x} \circ T^{-1}[A \times \mathbb{R}] - \mu^{\otimes(n-1)}[A] \right| = 0.$$

As explained in [2], this means that under $\mathbb{P}[\cdot | \bar{W}_n \in (x, x+1)]$, asymptotically one gets $n-1$ independent random variables after forgetting the largest jump.

The proof of Theorem 1 is based on the following invariance principle concerning a conditioned random walk with negative drift, which is a simple consequence of Proposition 2.1.

Proposition 2.2. *Let U be a uniformly distributed random variable on $[0, 1]$. Then the following convergence holds in $\mathbb{D}([0, 1], \mathbb{R})$:*

$$\left(\frac{W_{\lfloor nt \rfloor}}{n}, 0 \leq t \leq 1 \mid W_n = -1 \right) \xrightarrow[n \rightarrow \infty]{(d)} (-\gamma t + \gamma \mathbb{1}_{U \leq t}, 0 \leq t \leq 1). \quad (2)$$

Proof. By the definition of \overline{W} , it is sufficient to check that the following convergence holds in $\mathbb{D}([0, 1], \mathbb{R})$:

$$\left(\frac{\overline{W}_{\lfloor nt \rfloor}}{n}, 0 \leq t \leq 1 \mid \overline{W}_n = \gamma n - 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\gamma \mathbb{1}_{U \leq t}, 0 \leq t \leq 1), \quad (3)$$

where U is a uniformly distributed random variable on $[0, 1]$. Denote by V_n the coordinate of the first maximal component of $(\overline{X}_1, \dots, \overline{X}_n)$. Set $\widetilde{W}_0 = 0$ and for $1 \leq i \leq n - 1$ set:

$$\widetilde{W}_i = \begin{cases} \overline{X}_1 + \overline{X}_2 + \dots + \overline{X}_i & \text{if } i < V_n \\ \overline{X}_1 + \overline{X}_2 + \dots + \overline{X}_{V_n-1} + \overline{X}_{V_n+1} + \dots + \overline{X}_{i+1} & \text{otherwise.} \end{cases}$$

By Proposition 2.1, for every $\epsilon > 0$:

$$\lim_{n \rightarrow \infty} \left| \mathbb{P} \left[\forall t \in [0, 1], \left| \frac{\widetilde{W}_{\lfloor (n-1)t \rfloor}}{n-1} \right| \leq \epsilon \mid \overline{W}_n = \gamma n - 1 \right] - \mathbb{P} \left[\forall t \in [0, 1], \left| \frac{\overline{W}_{\lfloor (n-1)t \rfloor}}{n-1} \right| \leq \epsilon \right] \right| = 0. \quad (4)$$

We now claim that for every $\epsilon > 0$:

$$\mathbb{P} \left[\forall t \in [0, 1], \left| \frac{\overline{W}_{\lfloor (n-1)t \rfloor}}{n-1} \right| \leq \epsilon \right] \xrightarrow[n \rightarrow \infty]{} 1. \quad (5)$$

To establish (5), note that $(|\overline{W}_i|)_{i \geq 0}$ is a submartingale, so that Doob's maximal inequality entails:

$$\mathbb{P} \left[\max_{1 \leq i \leq n-1} |\overline{W}_i| \geq (n-1)\epsilon \right] \leq \frac{\mathbb{E} [|\overline{W}_{n-1}|]}{\epsilon(n-1)}.$$

By [31], $\mathbb{E} [|\overline{W}_{n-1}|] / (n-1) \rightarrow 0$ as $n \rightarrow \infty$. This shows (5).

Combining (5) with (4), we get the following convergence in $\mathbb{D}([0, 1], \mathbb{R})$:

$$\left(\frac{\widetilde{W}_{\lfloor (n-1)t \rfloor}}{n-1}; 0 \leq t \leq 1 \mid \overline{W}_n = \gamma n - 1 \right) \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \mathbf{0}, \quad (6)$$

where $\mathbf{0}$ stands for the constant function equal to 0 on $[0, 1]$. In addition, note that on the event $\{\overline{W}_n = \gamma n - 1\}$, we have $\overline{X}_{V_n} = \gamma n - 1 - \widetilde{W}_{\lfloor n-1 \rfloor}$. The following joint convergence in distribution thus holds in $\mathbb{D}([0, 1], \mathbb{R}) \otimes \mathbb{R}$:

$$\left(\left(\frac{\widetilde{W}_{\lfloor (n-1)t \rfloor}}{n-1}; 0 \leq t \leq 1 \right), \frac{\overline{X}_{V_n}}{n} \mid \overline{W}_n = \gamma n - 1 \right) \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} (\mathbf{0}, \gamma). \quad (7)$$

Standard properties of the Skorokhod topology then show that the following convergence holds in $\mathbb{D}([0, 1], \mathbb{R})$:

$$\left(\frac{\overline{W}_{\lfloor nt \rfloor}}{n} - \frac{\overline{X}_{V_n}}{n} \mathbb{1}_{\{t \geq \frac{V_n}{n}\}}; 0 \leq t \leq 1 \mid \overline{W}_n = \gamma n - 1 \right) \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \mathbf{0}. \quad (8)$$

Next, note that the convergence (7) implies that under $\mathbb{P} [\cdot \mid \overline{W}_n = \gamma n - 1]$, $(\overline{X}_1, \dots, \overline{X}_n)$ has a unique maximal component with probability tending to one as $n \rightarrow \infty$. Since the distribution of

$(\bar{X}_1, \dots, \bar{X}_n)$ under $\mathbb{P}[\cdot | \bar{W}_n = \gamma n - 1]$ is cyclically exchangeable, one easily gets that the law of V_n/n under $\mathbb{P}[\cdot | \bar{W}_n = \gamma n - 1]$ converges to the uniform distribution on $[0, 1]$. Also from (7) we know that \bar{X}_{V_n}/n under $\mathbb{P}[\cdot | \bar{W}_n = \gamma n - 1]$ converges in probability to γ . It follows that

$$\left(\frac{\bar{X}_{V_n}}{n} \mathbb{1}_{\{\frac{V_n}{n} \leq t\}}, 0 \leq t \leq 1 \middle| \bar{W}_n = \gamma n - 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\gamma \mathbb{1}_{U \leq t}, 0 \leq t \leq 1), \quad (9)$$

where U is uniformly over $[0, 1]$. Since (8) holds in probability, we can combine (8) and (9) to get (3). This completes the proof. \square

Using completely different techniques, a result similar to Proposition 2.2 has been proved by Durrett [12, Theorem 3.2] when the conditioning by the event $\{W_n = -1\}$ is replaced by the conditioning by the event $\{W_n > -1\}$. However, the so-called local conditioning by $\{W_n = -1\}$ is often more difficult to analyze (see e.g. [2]).

Before proving Theorem 1, we need to introduce some notation. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$, set $\mathcal{M}(\mathbf{x}) = \max_{1 \leq i \leq n} x_i$. Recall the notation $\mathbf{V}(\mathbf{x})$ for the Vervaat transform of \mathbf{x} . Note that $\mathcal{M}(\mathbf{x}) = \mathcal{M}(\mathbf{V}(\mathbf{x}))$. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded continuous function. Recall that $\Delta(\mathbf{t}_n)$ denotes the maximal out-degree of a vertex of \mathbf{t}_n . Since the maximal jump of $\mathcal{W}(\mathbf{t}_n)$ is equal to $\Delta(\mathbf{t}_n) - 1$, it follows from the remark following Proposition 1.6 that:

$$\begin{aligned} \mathbb{E}[F(\Delta(\mathbf{t}_n))] &= \mathbb{E}[F(\mathcal{M}(\mathbf{V}(X_1, X_2, \dots, X_n)) + 1) | W_n = -1] \\ &= \mathbb{E}[F(\mathcal{M}(X_1, X_2, \dots, X_n) + 1) | W_n = -1] \end{aligned} \quad (10)$$

Recall that since μ satisfies Assumption (H_θ) , \bar{W}_1 belongs to the domain of attraction of a spectrally positive strictly stable law of index $2 \wedge \theta$. Hence there exists a slowly varying function L such that $\bar{W}_n / (L(n)n^{1/(2 \wedge \theta)})$ converges in distribution toward Y_1 . We set $B_n = L(n)n^{1/(2 \wedge \theta)}$ and prove that Theorem 1 holds with this choice of B_n . The function L is not unique, but if \tilde{L} is another slowly function with the same property we have $L(n)/\tilde{L}(n) \rightarrow 1$ as $n \rightarrow \infty$. So our results do not depend on the choice of L . Note that when μ has finite variance σ^2 , one may take $B_n = \sigma\sqrt{n/2}$, and when $\mathcal{L} = c + o(1)$ one may choose L to be a constant function.

We are now ready to prove Theorem 1.

Proof of Theorem 1. If $Z \in \mathbb{D}([0, 1], \mathbb{R})$, denote by $\bar{\Delta}(Z) = \sup_{0 \leq s < 1} (Z_s - Z_{s-})$ the largest jump of Z . Since $\bar{\Delta} : \mathbb{D}([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ is continuous, from Proposition 2.2 we get that, under the conditioned probability measure $\mathbb{P}[\cdot | W_n = -1]$, $\mathcal{M}(X_1, X_2, \dots, X_n)/n$ converges in probability toward γ as $n \rightarrow \infty$. Assertion (i) immediately follows from (10).

For the second assertion, keeping the notation of the proof of Proposition 2.2, we get by Proposition 2.1 that for every bounded continuous function $F : \mathbb{D}([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$

$$\lim_{n \rightarrow \infty} \left| \mathbb{E} \left[F \left(\frac{\tilde{W}_{\lfloor (n-1)t \rfloor}}{B_n}; 0 \leq t \leq 1 \right) \middle| \bar{W}_n = \gamma n - 1 \right] - \mathbb{E}[F(Y_t, 0 \leq t \leq 1)] \right| = 0. \quad (11)$$

The second assertion follows, since the jumps of $(\tilde{W}_{\lfloor (n-1)t \rfloor}; 0 \leq t \leq 1)$ have the same distribution as the out-degrees, minus one, of all the vertices of \mathbf{t}_n , except $u_\star(\mathbf{t}_n)$.

For (iii), if V_n is as in the proof of Proposition 2.2, note that we have

$$\mathcal{M}(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n) = \bar{X}_{V_n} = \gamma n - 1 - \sum_{i \neq V_n} \bar{X}_i = \gamma n - 1 - \tilde{W}_{n-1}$$

on the event $\{\overline{W}_n = \gamma n - 1\}$. As noted in [2, Formula (2.7)], it follows from (11) that

$$\left(\frac{\mathcal{M}(\overline{X}_1, \overline{X}_2, \dots, \overline{X}_n) - \gamma n}{B_n} \middle| \overline{W}_n = \gamma n - 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} -Y_1.$$

Since $\mathcal{M}(\overline{X}_1, \overline{X}_2, \dots, \overline{X}_n) = \mathcal{M}(X_1, X_2, \dots, X_n) + \gamma$, we thus get that

$$\left(\frac{\mathcal{M}(X_1, X_2, \dots, X_n) + 1 - \gamma n}{B_n} \middle| W_n = -1 \right) \xrightarrow[n \rightarrow \infty]{(d)} -Y_1.$$

Assertion (iii) then immediately follows from (10). This completes the proof. \square

Remark 2.3. The preceding proof shows that assertion (i) in Theorem 1 remain true when μ is subcritical and both (1) and Proposition 2.1 hold. These conditions are more general than those of Assumption (H_θ) : see e.g. [10, Section 9] for examples of probability distributions that do not satisfy Assumption (H_θ) but such that (1) holds. Note that assertions (ii) and (iii) in Theorem 1 rely on the fact that μ belongs to the domain of attraction of a stable law. Note also that there exist subcritical probability distributions such that none of the assertions of Theorem 1 hold (see [18, Example 19.37] for an example).

Recall that $U(\tau)$ is the index in the lexicographical order of the first vertex of \mathbf{t}_n with maximal out-degree. We will need the following useful consequence of Theorem 1:

Corollary 2.4. *Fix $c \in (0, \gamma)$. We have $U(\mathbf{t}_n) = \min\{j \geq 0; \mathcal{W}_{j+1}(\mathbf{t}_n) - \mathcal{W}_j(\mathbf{t}_n) \geq cn\}$ with probability tending to one as $n \rightarrow \infty$.*

This is an immediate consequence of Theorem 1. Indeed, assertions (i) and (ii) of the latter theorem imply that for every $c \in (0, \gamma)$, there is a unique vertex of \mathbf{t}_n with out-degree at least cn , with probability tending to one as $n \rightarrow \infty$.

By applying the Vervaat transformation in continuous time to the convergence of Proposition 2.2, standard properties of the Skorokhod topology imply the following invariance principle for the Lukasiewicz path coding \mathbf{t}_n (we leave details to the reader since we will not need this result later). See Fig. 4 for a simulation.

Proposition 2.5. *The following assertions hold.*

(i) *We have:*

$$\sup_{0 \leq i \leq U(\mathbf{t}_n)} \frac{\mathcal{W}_i(\mathbf{t}_n)}{n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0.$$

(ii) *The following convergence holds in distribution in $\mathbb{D}([0, 1], \mathbb{R})$:*

$$\left(\frac{\mathcal{W}_{\lfloor nt \rfloor \vee (U(\mathbf{t}_n)+1)}(\mathbf{t}_n)}{n}, 0 \leq t \leq 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\gamma(1-t), 0 \leq t \leq 1).$$

Using completely different techniques, a similar result has been proved in the context of conditioned random walks by Durrett [12, Theorem 3.2] when \mathbf{t}_n is a GW_μ conditioned on having at least n vertices and in addition μ has finite variance. However, the so-called local conditioning by having a fixed number of vertices is often more difficult to analyze (see e.g. [11, 29]). Property (i) shows that $(\mathcal{W}_{\lfloor nt \rfloor}(\mathbf{t}_n)/n, 0 \leq t \leq 1)$ does not converge in distribution in $\mathbb{D}([0, 1], \mathbb{R})$ toward $(\gamma(1-t), 0 \leq t \leq 1)$ and this explains why we look at the Lukasiewicz path only after time $U(\mathbf{t}_n)$ in (ii).

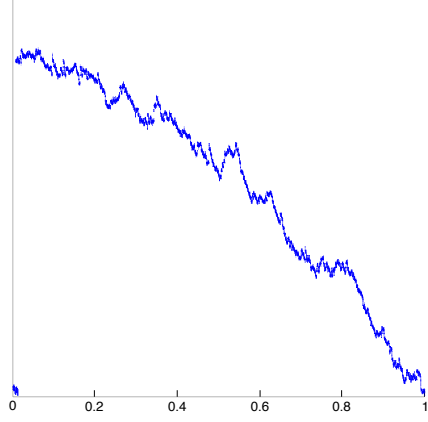


Figure 4: A simulation of a Lukasiewicz path of a large non-generic tree.

2.2 Location of the vertex with maximal out-degree

Our goal is to prove Theorem 2. We first need to introduce some notation. It is well known that the mean number of vertices of a GW_μ tree at generation n is \mathbf{m}^n . As a consequence, we have $\mathbb{E}_\mu[\zeta(\tau)] = 1 + \mathbf{m} + \mathbf{m}^2 + \cdots = 1/(1 - \mathbf{m}) = 1/\gamma$. Moreover, for $n \geq 1$, by Kemperman's formula (see e.g. [30, Section 5]):

$$\mathbb{P}_\mu[\zeta(\tau) = n] = \frac{1}{n} \mathbb{P}[W_n = -1] = \frac{1}{n} \mathbb{P}[\overline{W}_n = \gamma n - 1] \underset{n \rightarrow \infty}{\sim} \frac{\mathcal{L}(n)}{(\gamma n)^{1+\theta}}, \quad (12)$$

where we have used (1) for the last estimate. It follows that the total progeny of a GW_μ tree belongs to the domain of attraction of a spectrally positive strictly stable law of index $2 \wedge \theta$. Hence we can find a slowly varying function L' such that the law of $(\zeta(\mathbf{f}) - n/\gamma) / (L'(n)n^{1/(2 \wedge \theta)})$ under $\mathbb{P}_{\mu,n}$ converges as $n \rightarrow \infty$ to the law of Y_1 , where we recall that $\mathbb{P}_{\mu,j}$ is the law of a forest of j independent GW_μ trees. We set $B'_n = L'(n)n^{1/(2 \wedge \theta)}$. The local limit theorem (see [15, Theorem 4.2.1]) implies that if we set $\varphi_j(k) = \mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) = k]$ for $j \geq 1$ and $k \geq 0$ (with the convention $\varphi_j(k) = 0$ for $k < 0$), then

$$\lim_{m \rightarrow \infty} \sup_{k \in \mathbb{Z}} \left| B'_m \varphi_m(k) - p_1 \left(\frac{k - m/\gamma}{B'_m} \right) \right| = 0, \quad (13)$$

where p_1 is the density of Y_1 . It is well known that p_1 is a bounded continuous function (see e.g. [34, I. 4]).

The following technical result establishes a useful link between B_n and B'_n .

Lemma 2.6. *We have $B'_n/B_n \rightarrow 1/\gamma^{1+1/(2 \wedge \theta)}$ as $n \rightarrow \infty$.*

The proof of Lemma 2.6 is postponed to the end of this section.

Proof of Theorem 2 (i). Fix $\epsilon > 0$ and an integer $i_0 \geq 0$. By Theorem 1 (iii) and Lemma 2.6, we may choose $A > 0$ such that:

$$|\mathbb{P}[U(\mathbf{t}_n) = i_0] - \mathbb{P}[U(\mathbf{t}_n) = i_0, |k_{i_0}(\mathbf{t}_n) - \gamma n| \leq AB'_n]| \leq \epsilon \quad (14)$$

By Theorem 1, for n large enough:

$$|\mathbb{P}[U(\mathbf{t}_n) = i_0, |k_{i_0}(\mathbf{t}_n) - \gamma n| \leq AB'_n] - \mathbb{P}[|k_{i_0}(\mathbf{t}_n) - \gamma n| \leq AB'_n]| \leq \epsilon. \quad (15)$$

By Proposition 1.4, we have for n large enough:

$$\begin{aligned} & \mathbb{P}[|k_{i_0}(\mathbf{t}_n) - \gamma n| \leq AB'_n] \\ &= \frac{1}{\mathbb{P}_\mu[\zeta(\tau) = n]} \sum_{|j - \gamma n| \leq AB'_n} \mathbb{P}[X_{i_0+1} = j - 1, \zeta = n] \\ &= \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}[X_{i_0+1} = j - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}[\mathbb{1}_{\{\forall m \leq i_0, W_m \geq 0\}} \varphi_{W_{i_0}+j}(n - i_0 - 1)], \end{aligned} \quad (16)$$

where we have used the Markov property of the random walk W at time $i_0 + 1$ for the last equality. To simplify notation, set $j_n(u) = \lfloor \gamma n + uB'_n \rfloor$ for $u \in \mathbb{R}$. Then write:

$$\begin{aligned} & \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}[X_{i_0+1} = j - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}[\mathbb{1}_{\{\forall m \leq i_0, W_m \geq 0\}} \varphi_{W_{i_0}+j}(n - i_0 - 1)] \\ &= \mathbb{E} \left[\mathbb{1}_{\{\forall m \leq i_0, W_m \geq 0\}} \int_{-AB'_n - o(1)}^{AB'_n + o(1)} du \frac{\mathbb{P}[X_1 = \lfloor \gamma n + u \rfloor - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \varphi_{W_{i_0} + \lfloor \gamma n + u \rfloor}(n - i_0 - 1) \right] \\ &= B'_n \mathbb{E} \left[\mathbb{1}_{\{\forall m \leq i_0, W_m \geq 0\}} \int_{-A - o(1)}^{A + o(1)} du \frac{\mathbb{P}[X_1 = j_n(u) - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \varphi_{W_{i_0} + j_n(u)}(n - i_0 - 1) \right] \end{aligned} \quad (17)$$

By using the dominated convergence theorem, we will now show that for a fixed integer $k \geq 0$

$$B'_n \int_{-A - o(1)}^{A + o(1)} du \frac{\mathbb{P}[X_1 = j_n(u) - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \varphi_{k + j_n(u)}(n - i_0 - 1) \xrightarrow{n \rightarrow \infty} \int_{-A}^A \frac{du}{\gamma^{1/(2 \wedge \theta)}} p_1 \left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2 \wedge \theta)}} \right) \quad (18)$$

We will then apply (18) with $k = W_{i_0}$. Since μ satisfies assumption (ii) in (H_θ) , we have $\mathbb{P}[X_1 = j_n(u) - 1] = \mathcal{L}(j_n(u))/(j_n(u))^{1+\theta}$. From (12), it follows that

$$\sup_{-A-1 \leq u \leq A+1} \left| \frac{\mathbb{P}[X_1 = j_n(u) - 1]}{\mathbb{P}_\mu[\zeta(\tau) = n]} - 1 \right| \xrightarrow{n \rightarrow \infty} 0 \quad (19)$$

Next, note that $B'_n/B'_{k+j_n(u)} \rightarrow 1/\gamma^{1/(2 \wedge \theta)}$ as $n \rightarrow \infty$, uniformly in $u \in (-A - 1, A + 1)$. From (13), it follows that

$$B'_n \varphi_{k+j_n(u)}(n - i_0 - 1) - \frac{1}{\gamma^{1/(2 \wedge \theta)}} p_1 \left(\frac{n - i_0 - 1 - \frac{1}{\gamma}(k + \lfloor \gamma n + uB'_n \rfloor)}{B'_{k+j_n(u)}} \right) \xrightarrow{n \rightarrow \infty} 0.$$

Hence

$$B'_n \varphi_{k+j_n(u)}(n - i_0 - 1) \xrightarrow{n \rightarrow \infty} \frac{1}{\gamma^{1/(2 \wedge \theta)}} p_1 \left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2 \wedge \theta)}} \right).$$

In addition, by (13), there exists a constant $C > 0$ which is independent of k and such that $0 \leq B'_n \varphi_{k+j_n(u)}(n - i_0 - 1) \leq C$ for every n sufficiently large and $u \in (-A - 1, A + 1)$. The convergence (18) then follows from an application of the dominated convergence theorem.

Another application of the dominated convergence theorem gives that the expression appearing in (17) converges toward

$$\mathbb{P}[\forall m \leq i_0, W_m \geq 0] \int_{-A}^A \frac{du}{\gamma^{1/(2 \wedge \theta)}} p_1 \left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2 \wedge \theta)}} \right).$$

as $n \rightarrow \infty$. It follows from (14), (15) and (16) that for n sufficiently large:

$$\left| \mathbb{P}[U(\mathbf{t}_n) = i_0] - \mathbb{P}[\forall m \leq i_0, W_m \geq 0] \int_{-A}^A \frac{du}{\gamma^{1/(2 \wedge \theta)}} p_1 \left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2 \wedge \theta)}} \right) \right| \leq 3\epsilon.$$

Finally, since p_1 is the density of a probability distribution, we have:

$$\lim_{A \rightarrow \infty} \int_{-A}^A \frac{du}{\gamma^{1/(2 \wedge \theta)}} p_1 \left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2 \wedge \theta)}} \right) = \int_{-\infty}^{\infty} \frac{du}{\gamma^{1/(2 \wedge \theta)}} p_1 \left(-\frac{1}{\gamma} \cdot \frac{u}{\gamma^{1/(2 \wedge \theta)}} \right) = \gamma.$$

Thus, by choosing A sufficiently large, we get that for n sufficiently large

$$|\mathbb{P}[U(\mathbf{t}_n) = i_0] - \gamma \mathbb{P}[\forall m \leq i_0, W_m \geq 0]| \leq 4\epsilon.$$

Since this holds for every $\epsilon > 0$, we get $\lim_{n \rightarrow \infty} \mathbb{P}[U(\mathbf{t}_n) = i_0] = \gamma \mathbb{P}[\forall m \leq i_0, W_m \geq 0]$. By Proposition 1.4, we have $\mathbb{P}[\forall m \leq i_0, W_m \geq 0] = \mathbb{P}_\mu[\zeta(\tau) \geq i_0 + 1]$. This completes the proof of assertion (i) in Theorem 2. \square

To prove the second assertion of Theorem 2, we will need the size-biased distribution associated with μ , which is the distribution of the random variable ζ^* such that:

$$\mathbb{P}[\zeta^* = k] := \frac{k\mu_k}{\mathfrak{m}} \quad k = 0, 1, \dots$$

The following result concerning the local convergence of \mathbf{t}_n as $n \rightarrow \infty$ will be useful. We refer the reader to [18, Section 6] for definitions and background concerning local convergence of trees (note that we need to consider trees that are not locally finite, so that this is slightly different from the usual setting).

Proposition 2.7 (Jonsson & Stefánsson [21], Janson [18]). *Let $\hat{\mathcal{T}}$ be the infinite random tree constructed as follows. Start with a spine composed of a random number S of vertices, where S is defined by:*

$$\mathbb{P}[S = i] = (1 - \mathfrak{m})\mathfrak{m}^{i-1}, \quad i = 1, 2, \dots \quad (20)$$

Then attach further branches as follows (see also Figure 5 below). At the top of the spine, attach an infinite number of branches, each branch being a GW_μ tree. At all the other vertices of the spine, a random number of branches distributed as $\zeta^ - 1$ is attached to either to the left or to the right of the spine, each branch being a GW_μ tree. At a vertex of the spine where k new branches are attached, the number of new branches attached to the left of the spine is uniformly distributed on $\{0, \dots, k\}$. Moreover all random choices are independent.*

Then \mathbf{t}_n converges locally in distribution toward $\hat{\mathcal{T}}$ as $n \rightarrow \infty$.

Proof of Theorem 2 (ii) and (iii). By Skorokhod's representation theorem (see e.g. [5, Theorem 6.7]) we can suppose that the convergence $\mathbf{t}_n \rightarrow \hat{\mathcal{T}}$ as $n \rightarrow \infty$ holds almost surely for the local

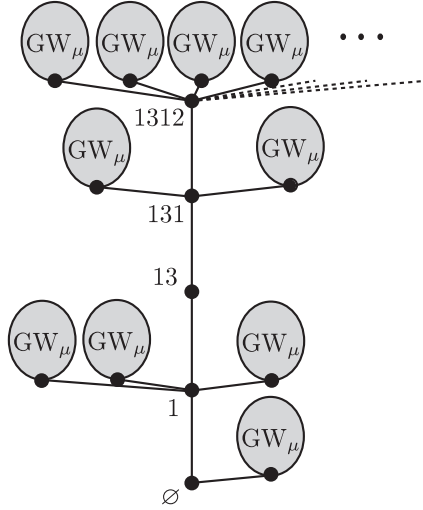


Figure 5: An illustration of $\hat{\mathcal{T}}$. Here, the spine is composed of the vertices $\emptyset, 1, 13, 13, 131, 1312$.

topology. Let $u_\star \in \hat{\mathcal{T}}$ be the vertex of the spine with largest generation. By (20), we have for $i \geq 0$:

$$\mathbb{P}[|u_\star| = i] = (1 - \mathbf{m})\mathbf{m}^i. \quad (21)$$

Recall the notation $U(\mathbf{t}_n)$ for the index of $u_\star(\mathbf{t}_n)$. Let $\epsilon > 0$. By assertion (i) of Theorem 2, which was proved at the beginning of this section, we can fix an integer K such that, for every n , $\mathbb{P}[U(\mathbf{t}_n) \leq K] > 1 - \epsilon$. From the local convergence of \mathbf{t}_n to $\hat{\mathcal{T}}$ (and the properties of local convergence, see in particular Lemma 6.3 in [18]) we can easily verify that

$$\mathbb{P}[\{u_\star(\mathbf{t}_n) \neq u_\star\} \cap \{U(\mathbf{t}_n) \leq K\}] \xrightarrow{n \rightarrow \infty} 0.$$

We conclude that $\mathbb{P}[u_\star(\mathbf{t}_n) \neq u_\star] \rightarrow 0$ as $n \rightarrow \infty$. Assertion (ii) of Theorem 2 now follows from (21). Similarly, the tree \mathbf{t}_n cut at $u_\star(\mathbf{t}_n)$ converges locally in distribution toward the tree $\hat{\mathcal{T}}$ cut at u_\star , giving assertion (iii). \square

Note that assertion (i) in Theorem 2 was needed to prove assertions (ii) and (iii). Indeed, the local convergence of \mathbf{t}_n toward $\hat{\mathcal{T}}$ would not have been sufficient to get that $\mathbb{P}[u_\star(\mathbf{t}_n) \neq u_\star] \rightarrow 0$.

We conclude this section by proving Lemma 2.6.

Proof of Lemma 2.6. Let σ^2 be the variance of μ . Note that $\sigma^2 = \infty$ if $\theta \in (1, 2)$, $\sigma^2 < \infty$ if $\theta > 2$ and that we can have either $\sigma^2 = \infty$ or $\sigma^2 < \infty$ for $\theta = 2$. When $\sigma^2 = \infty$, the desired result follows from classical results expressing B_n in terms of μ . Indeed, in the case $\theta < 2$, we may choose B_n and B'_n such that (see e.g. [26, Theorem 1.10]):

$$\frac{B'_n}{B_n} = \frac{\inf \{x \geq 0; \mathbb{P}_\mu[\zeta(\tau) \geq x] \leq \frac{1}{n}\}}{\inf \{x \geq 0; \mu([x, \infty)) \leq \frac{1}{n}\}}.$$

Property (ii) in Assumption (H_θ) and (12) entail that $\mathbb{P}_\mu[\zeta(\tau) \geq x]/\mu([x, \infty)) \rightarrow 1/\gamma^{1+\theta}$ as $x \rightarrow \infty$. The result easily follows. The case when $\sigma^2 = \infty$ and $\theta = 2$ is treated by using similar arguments. We leave details to the reader.

We now concentrate on the case $\sigma^2 < \infty$. Note that necessarily $\theta \geq 2$. Let σ'^2 be the variance of $\zeta(\tau)$ under \mathbb{P}_μ (from (12) this variance is finite when $\sigma^2 < \infty$). We shall show that $\sigma' = \sigma/\gamma^{3/2}$. The desired result will then follow by the classical central limit theorem since we may take $B_n = \sigma\sqrt{n/2}$ and $B'_n = \sigma'\sqrt{n/2}$. In order to calculate σ'^2 , we introduce the Galton-Watson process $(\mathfrak{Z}_i)_{i \geq 0}$ with offspring distribution μ such that $\mathfrak{Z}_0 = 1$. Recall that $\mathbb{E}[\mathfrak{Z}_i] = \mathfrak{m}^i$. Then note that:

$$\sigma'^2 = \mathbb{E}_\mu [\zeta(\tau)^2] - \mathbb{E}_\mu [\zeta(\tau)]^2 = \mathbb{E} \left[\left(\sum_{i=0}^{\infty} \mathfrak{Z}_i \right)^2 \right] - \frac{1}{\gamma^2}.$$

Since $(\mathfrak{Z}_i/\mathfrak{m}^i)_{i \geq 0}$ is a martingale with respect to the filtration generated by $(\mathfrak{Z}_i)_{i \geq 0}$, we have $\mathbb{E}[Z_i Z_j] = \mathfrak{m}^{j-i} \mathbb{E}[Z_i^2]$. Also using the well-known fact that for $i \geq 1$ the variance of \mathfrak{Z}_i is $\sigma^2 \mathfrak{m}^{i-1}(\mathfrak{m}^i - 1)/(\mathfrak{m} - 1)$ (see e.g. [3, Section 1.2]), write:

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i=0}^{\infty} \mathfrak{Z}_i \right)^2 \right] &= \sum_{i=0}^{\infty} \mathbb{E}[\mathfrak{Z}_i^2] + 2 \sum_{0 \leq i < j} \mathfrak{m}^{j-i} \mathbb{E}[\mathfrak{Z}_i^2] = \sum_{i=0}^{\infty} \mathbb{E}[\mathfrak{Z}_i^2] \left(1 + \frac{2\mathfrak{m}}{1 - \mathfrak{m}} \right) \\ &= \left(1 + \sum_{i=1}^{\infty} \left(\frac{\sigma^2 \mathfrak{m}^{i-1}(\mathfrak{m}^i - 1)}{\mathfrak{m} - 1} + \mathfrak{m}^{2i} \right) \right) \left(1 + \frac{2\mathfrak{m}}{1 - \mathfrak{m}} \right) \\ &= \frac{\sigma^2}{\gamma^3} + \frac{1}{\gamma^2}. \end{aligned}$$

This entails $\sigma' = \sigma/\gamma^{3/2}$ and the conclusion follows. \square

2.3 Cutting the tree at the vertex with maximum out-degree

To analyze the structure of \mathfrak{t}_n we shall cut \mathfrak{t}_n into two subtrees, one being the subtree of the descendants $u_\star(\mathfrak{t}_n)$ and the other being the subtree formed by the vertices which are not descendants of $u_\star(\mathfrak{t}_n)$. A major difficulty is that these two subtrees are not independent. To overcome this difficulty, the idea is to use a different conditioning: we shall reduce the study of a tree conditioned on having total size n to the study of a tree conditioned by the event that its first vertex with strictly more than $\gamma n/2$ children has a fixed out-degree. The main advantage is that under this conditioning the two subtrees become independent.

We need to introduce some important notation which will be used throughout this work. If τ is a tree, let $u_\star^{(n)}(\tau)$ be the first vertex of τ in lexicographical order with strictly more than $\gamma n/2$ children (with the convention $u_\star^{(n)}(\tau) = \partial$ if there is no such vertex, where ∂ is a cemetery point). Let $\kappa_n(\tau)$ be the number of children of $u_\star^{(n)}(\tau)$ (with the convention $\kappa_n(\tau) = 0$ if $u_\star^{(n)}(\tau) = \partial$). Recall the notation $T_u \tau$ for the tree shifted at u . For $1 \leq j \leq \kappa_n(\tau)$, let $u_j^{(n)}(\tau)$ denote the j -th child of $u_\star^{(n)}(\tau)$ and set $\tau_j^{(n)} = T_{u_j^{(n)}}(\tau)$. For $1 \leq i \leq j \leq \kappa_n(\tau)$, we set $\mathcal{F}_{i,j}^{(n)} = \mathcal{F}_{i,j}^{(n)}(\tau) = (\tau_i^{(n)}, \dots, \tau_j^{(n)})$. By definition, a pointed tree is a tree with a distinguished vertex. Finally, if $u \in \tau$, let $\text{Cut}_u(\tau)$ be the tree obtained from τ by removing all the descendants of u and pointed at u . To simplify notation, set $\text{Cut}^{(n)}(\tau) = \text{Cut}_{u_\star^{(n)}(\tau)}(\tau)$, with the convention $\text{Cut}^{(n)}(\tau) = \partial$ if $u_\star^{(n)}(\tau) = \partial$ (that is if there is no vertex of τ with strictly more than $\gamma n/2$ children). Recall the notation $\mathbb{P}_{\mu,j}$ for the law of a forest of j independent GW $_\mu$ trees. The following result easily follows from the properties of Galton-Watson trees:

Lemma 2.8. *For every $j \geq 1$ such that $\mathbb{P}_\mu[\kappa_n(\tau) = j] > 0$, under the conditional probability measure $\mathbb{P}_\mu[\cdot | \kappa_n = j]$, the random variables $\text{Cut}^{(n)}$ and $\mathcal{F}_{1,j}^{(n)}$ are independent and, in addition, $\mathcal{F}_{1,j}^{(n)}$ is distributed according to $\mathbb{P}_{\mu,j}$ and $\text{Cut}^{(n)}$ has the same distribution as $\text{Cut}^{(n)}$ under the conditioned probability measure $\mathbb{P}_\mu[\cdot | \text{Cut}^{(n)} \neq \partial]$.*

We next show that under the conditioned probability measure $\mathbb{P}_\mu[\cdot | \text{Cut}^{(n)} \neq \partial]$, the tree $\text{Cut}^{(n)}$ converges locally in distribution toward $\text{Cut}_{u_*} \hat{\mathcal{T}}$, where $\hat{\mathcal{T}}$ is the infinite tree appearing in Proposition 2.7.

Lemma 2.9. *Under the conditioned probability measure $\mathbb{P}_\mu[\cdot | \text{Cut}^{(n)} \neq \partial]$, $\text{Cut}^{(n)}$ converges locally in distribution toward $\text{Cut}_{u_*} \hat{\mathcal{T}}$ as $n \rightarrow \infty$.*

Note that Lemma 2.9 is not a simple consequence of Theorem 2 (iii) since the events involved in the conditionings are not the same. Before we proceed to the proof of Lemma 2.9, we state a technical lemma.

Lemma 2.10. *Set $j_n(u) = \lfloor \gamma n + uB'_n \rfloor$ for $u \in \mathbb{R}$. Then for every $A > 0$:*

$$\lim_{n \rightarrow \infty} \sup_{u \in [-A-1, A+1]} \left| \frac{\mathbb{P}_\mu[\kappa_n(\tau) = j_n(u)]}{\mathbb{P}_\mu[\zeta(\tau) = n]} - \frac{1}{\gamma} \right| = 0.$$

Proof. By Proposition 1.4, for n sufficiently large, we have for every $u \in [-A-1, A+1]$:

$$\begin{aligned} \mathbb{P}_\mu[\kappa_n(\tau) = j_n(u)] &= \sum_{i=1}^{\infty} \mathbb{P} \left[X_1 < \frac{\gamma n}{2}, \dots, X_{i-1} < \frac{\gamma n}{2}, X_i = j_n(u) - 1, W_m \geq 0 \text{ for } 1 \leq m \leq i-1 \right] \\ &= \mathbb{P}[X_1 = j_n(u) - 1] \sum_{i=1}^{\infty} \mathbb{P} \left[X_1 < \frac{\gamma n}{2}, \dots, X_{i-1} < \frac{\gamma n}{2}, W_m \geq 0 \text{ for } 1 \leq m \leq i-1 \right]. \end{aligned}$$

The monotone convergence theorem entails:

$$\frac{\mathbb{P}_\mu[\kappa_n(\tau) = j_n(u)]}{\mathbb{P}[X_1 = j_n(u) - 1]} \xrightarrow{n \rightarrow \infty} \sum_{i=1}^{\infty} \mathbb{P}[W_m \geq 0 \text{ for } 1 \leq m \leq i-1],$$

uniformly in $u \in [-A-1, A+1]$. By Proposition 1.4, the last sum is equal to :

$$\sum_{i=1}^{\infty} \mathbb{P}_\mu[\zeta(\tau) \geq i] = \mathbb{E}_\mu[\zeta(\tau)] = \frac{1}{\gamma}$$

In addition, (19) gives that $\mathbb{P}[X_1 = j_n(u) - 1] / \mathbb{P}_\mu[\zeta(\tau) = n]$ converges toward 1 as $n \rightarrow \infty$, uniformly in $u \in [-A-1, A+1]$. The conclusion immediately follows. \square

Proof of Lemma 2.9. Fix a tree τ_0 pointed at a leaf. By Theorem 2 (iii), $\text{Cut}_{u_*(\mathbf{t}_n)}(\mathbf{t}_n)$ converges locally in distribution toward $\text{Cut}_{u_*} \hat{\mathcal{T}}$ as $n \rightarrow \infty$. In addition, by Corollary 2.4, $u_*^{(n)}(\mathbf{t}_n) = u_*(\mathbf{t}_n)$ with probability tending to one as $n \rightarrow \infty$. It follows from Theorem 1 (iii) and Lemma 2.6 that $(\kappa_n(\mathbf{t}_n) - \gamma n) / B'_n$ converges in distribution as $n \rightarrow \infty$. We can thus choose $A > 0$ such that for every n sufficiently large:

$$\left| \mathbb{P}[\text{Cut}_{u_*(\mathbf{t}_n)}(\mathbf{t}_n) = \tau_0] - \mathbb{P}[\text{Cut}^{(n)}(\mathbf{t}_n) = \tau_0, |\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n] \right| \leq \epsilon. \quad (22)$$

It is thus sufficient to show that, for an appropriate choice A , for n sufficiently large we have

$$\left| \mathbb{P} [\text{Cut}^{(n)}(\mathbf{t}_n) = \tau_0, |\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n] - \mathbb{P}_\mu [\text{Cut}^{(n)}(\tau) = \tau_0 | \text{Cut}^{(n)} \neq \partial] \right| \leq 2\epsilon. \quad (23)$$

To this end, write

$$\begin{aligned} & \mathbb{P} [\text{Cut}^{(n)}(\mathbf{t}_n) = \tau_0, |\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n] \\ &= \sum_{|j-\gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu [\kappa_n = j]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \mathbb{P}_\mu [\text{Cut}^{(n)}(\tau) = \tau_0, \zeta(\mathcal{F}_{1,j}) + \zeta(\tau_0) = n | \kappa_n = j] \\ &= \mathbb{P}_\mu [\text{Cut}^{(n)}(\tau) = \tau_0 | \text{Cut}^{(n)} \neq \partial] \sum_{|j-\gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu [\kappa_n = j]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \varphi_j(n - \zeta(\tau_0)), \end{aligned} \quad (24)$$

where we have used Lemma 2.8 for the last equality and recall the notation $\varphi_j(k) = \mathbb{P}_{\mu,j} [\zeta(\mathbf{f}) = k]$. Using Lemma 2.10, the same argument that led us to (18) shows that

$$\sum_{|j-\gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu [\kappa_n = j]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \varphi_j(n - \zeta(\tau_0)) \xrightarrow[n \rightarrow \infty]{} \int_{-A}^A du \frac{1}{\gamma^{1+1/(2\wedge\theta)}} p_1 \left(-\frac{u}{\gamma^{1+1/(2\wedge\theta)}} \right).$$

Since the term appearing in the right-hand side of the previous expression converges to 1 as $A \rightarrow \infty$, we may choose A sufficiently large so that

$$\left| \int_{-A}^A du \frac{1}{\gamma^{1+1/(2\wedge\theta)}} p_1 \left(-\frac{u}{\gamma^{1+1/(2\wedge\theta)}} \right) - 1 \right| \leq \epsilon.$$

This implies that for n sufficiently large

$$\left| \sum_{|j-\gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu [\kappa_n = j]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \varphi_j(n - \zeta(\tau_0)) - 1 \right| \leq 2\epsilon.$$

Then (23) follows from (24). This completes the proof. \square

2.4 Subtrees branching off the vertex with maximum out-degree

We now turn to the proof of Theorem 3. By Corollary 2.4, we have $u_\star^{(n)}(\mathbf{t}_n) = u_\star(\mathbf{t}_n)$ with probability tending to one as $n \rightarrow \infty$. It is thus sufficient to establish Theorem 3 when the process Z is replaced by the process $Z^{(n)}$ defined as follows. For every tree τ and for every $1 \leq j \leq \kappa_n(\tau)$, let $\zeta_j^{(n)}(\tau) = \zeta(\tau_j^{(n)})$ be the number of descendants of $\mathbf{u}_j^{(n)}(\tau)$, and set $Z_j^{(n)}(\tau) = \zeta_1^{(n)}(\tau) + \zeta_2^{(n)}(\tau) + \dots + \zeta_j^{(n)}(\tau)$. We need to prove that:

$$\left(\frac{Z_{\lfloor \kappa_n(\mathbf{t}_n)t \rfloor}^{(n)}(\mathbf{t}_n) - \kappa_n(\mathbf{t}_n)t/\gamma}{B_n}, 0 \leq t \leq 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} \left(\frac{1}{\gamma} Y_t, 0 \leq t \leq 1 \right). \quad (25)$$

Let $(\mathcal{Z}_i)_{i \geq 0}$ be the random walk which starts at 0 and whose jump distribution has the same law as the total progeny of a GW_μ tree. Note that $\mathbb{P} [\mathcal{Z}_j = k] = \mathbb{P}_{\mu,j} [\zeta(\mathbf{f}) = k]$. Recall from the beginning of Section 2.2 that the distribution of \mathcal{Z}_1 belongs to the domain of attraction of a spectrally positive strictly stable law of index $2 \wedge \theta$. In particular, if B'_n is defined as in the beginning of Section 2.2, the following convergence holds in distribution in the space $\mathbb{D}([0, 1], \mathbb{R})$:

$$\left(\frac{\mathcal{Z}_{\lfloor nt \rfloor} - nt/\gamma}{B'_n}, 0 \leq t \leq 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} (Y_t, 0 \leq t \leq 1). \quad (26)$$

Proof of Theorem 3. We shall show that for every fixed $\eta \in (0, 1)$:

$$\left(\frac{Z_{\lfloor \kappa_n(\mathbf{t}_n)t \rfloor}^{(n)}(\mathbf{t}_n) - \kappa_n(\mathbf{t}_n)t/\gamma}{B'_{\kappa_n(\mathbf{t}_n)}}, 0 \leq t \leq \eta \right) \xrightarrow[n \rightarrow \infty]{(d)} (Y_t, 0 \leq t \leq \eta). \quad (27)$$

Recalling Lemma 2.6, our claim (25) will follow from a time-reversal argument since the vectors $(\zeta_1^{(n)}(\mathbf{t}_n), \zeta_2^{(n)}(\mathbf{t}_n), \dots, \zeta_{\kappa_n(\mathbf{t}_n)}^{(n)}(\mathbf{t}_n))$ and $(\zeta_{\kappa_n(\mathbf{t}_n)}^{(n)}(\mathbf{t}_n), \zeta_{\kappa_n(\mathbf{t}_n)-1}^{(n)}(\mathbf{t}_n), \dots, \zeta_1^{(n)}(\mathbf{t}_n))$ have the same distribution. Tightness follows from the time-reversal argument and also continuity at $t = 1$.

Let $F : \mathbb{D}([0, \eta], \mathbb{R}) \rightarrow \mathbb{R}_+$ be a bounded continuous function and to simplify notation set, for every tree τ :

$$\widetilde{Z}^{(\kappa_n)}(\tau) = \left(\frac{Z_{\lfloor \kappa_n(\tau)t \rfloor}^{(n)}(\tau) - \kappa_n(\tau)t/\gamma}{B'_{\kappa_n(\tau)}}, 0 \leq t \leq \eta \right).$$

Fix $\epsilon > 0$. As in the proof of Lemma 2.9, we can choose $A > 0$ such that for every n sufficiently large:

$$\left| \mathbb{E}_\mu \left[F \left(\widetilde{Z}^{(\kappa_n)}(\tau) \right) \mid \zeta(\tau) = n \right] - \mathbb{E}_\mu \left[F \left(\widetilde{Z}^{(\kappa_n)}(\tau) \right) \mathbb{1}_{\{|\kappa_n(\tau) - \gamma n| \leq AB'_n\}} \mid \zeta(\tau) = n \right] \right| \leq \epsilon. \quad (28)$$

Now let $N_0^{(n)}(\tau) = \zeta(\tau) + 1 - \zeta(T_{u_\star^{(n)}\tau})$ denote the number of vertices of τ which are not strict descendants of $u_\star^{(n)}(\tau)$, so that $N_0^{(n)}(\tau) = \zeta(\text{Cut}^{(n)}(\tau))$. Without risk of confusion, in the sequel we write κ_n instead of $\kappa_n(\tau)$, $N_0^{(n)}$ instead of $N_0^{(n)}(\tau)$ and so on. Obviously $\{\zeta(\tau) = n\} = \{Z_{\kappa_n}^{(n)} + N_0^{(n)} = n\}$. To simplify notation, let $(Z_j)_{j \geq 0}$ and $\mathcal{N}_0^{(n)}$ be independent under \mathbb{P} and such that $(Z_j)_{j \geq 0}$ is distributed as explained before (26), and $\mathcal{N}_0^{(n)}$ has the same law as $N_0^{(n)}$ under $\mathbb{P}_\mu[\cdot \mid \text{Cut}^{(n)} \neq \emptyset]$. Finally, set for $k \geq 1$:

$$\mathcal{Z}^{(k)} = \left(\frac{Z_{\lfloor kt \rfloor} - kt/\gamma}{B'_k}, 0 \leq t \leq \eta \right).$$

By Lemma 2.8:

$$\begin{aligned} \mathbb{E}_\mu \left[F \left(\widetilde{Z}^{(\kappa_n)} \right) \mathbb{1}_{\{|\kappa_n - \gamma n| \leq AB'_n\}} \mid \zeta(\tau) = n \right] \\ = \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E} \left[F \left(\widetilde{Z}^{(j)} \right) \mathbb{1}_{\zeta(\mathcal{T}_{1,j}) + N_0^{(n)} = n} \mid \kappa_n = j \right] \\ = \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E} \left[F \left(\mathcal{Z}^{(j)} \right) \mathbb{1}_{\{Z_j + \mathcal{N}_0^{(n)} = n\}} \right]. \end{aligned} \quad (29)$$

Recall the notation $\varphi_j(k) = \mathbb{P}[Z_j = k]$. To simplify notation, for integers $j, n \geq 0$ we set $\mathcal{R}_j^{(n)} = \mathbb{E} \left[F \left(\mathcal{Z}^{(j)} \right) \mathbb{1}_{\{Z_j + \mathcal{N}_0^{(n)} = n\}} \right]$. It follows from the Markov property of the random walk $(Z_j)_{j \geq 0}$ applied at time $\lfloor \eta j \rfloor$ that:

$$\mathcal{R}_j^{(n)} = \mathbb{E} \left[F \left(\mathcal{Z}^{(j)} \right) \varphi_{j - \lfloor \eta j \rfloor}(n - \mathcal{N}_0^{(n)} - Z_{\lfloor \eta j \rfloor}) \right].$$

Recall that $j_n(u) = \lfloor \gamma n + uB'_n \rfloor$ for $u \in \mathbb{R}$. Then the sum appearing in (29) is equal to:

$$\int_{-AB'_n + o(1)}^{AB'_n + o(1)} du \frac{\mathbb{P}_\mu[\kappa_n = \lfloor \gamma n + u \rfloor]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathcal{R}_{\lfloor \gamma n + u \rfloor}^{(n)} = \int_{-A + o(1)}^{A + o(1)} du B'_n \frac{\mathbb{P}_\mu[\kappa_n = j_n(u)]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathcal{R}_{j_n(u)}^{(n)} \quad (30)$$

We shall now show that there exist $\alpha, \beta > 0$ such that :

$$\int_{-A+o(1)}^{A+o(1)} du B'_n \frac{\mathbb{P}_\mu [\kappa_n = j_n(u)]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \mathcal{R}_{j_n(u)}^{(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{\beta\gamma} \int_{-A}^A du \mathbb{E} \left[F((Y_t)_{0 \leq t \leq \eta}) p_1 \left(-\frac{u}{\beta\gamma} - \frac{Y_\eta}{\alpha} \right) \right], \quad (31)$$

where p_1 is the density of Y_1 . Recall that p_1 is a bounded continuous function.

We first show that there exist $\alpha, \beta > 0$ such that for fixed $u \in \mathbb{R}$:

$$B'_n \mathcal{R}_{j_n(u)}^{(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{\beta} \mathbb{E} \left[F((Y_t)_{0 \leq t \leq \eta}) p_1 \left(-\frac{u}{\beta\gamma} - \frac{Y_\eta}{\alpha} \right) \right]. \quad (32)$$

To this end, we start by establishing a few useful convergences. Set $\alpha = (1 - \eta)^{1/(2 \wedge \theta)}$. By (26) and Lemma 2.9, we have the following joint convergence in distribution:

$$\left(\mathcal{Z}^{(j_n(u))}, \frac{\mathcal{Z}_{\lfloor j_n(u)\eta \rfloor} - j_n(u)\eta/\gamma}{B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor}}, \mathcal{N}_0^{(n)} \right) \xrightarrow[n \rightarrow \infty]{(d)} \left((Y_t)_{0 \leq t \leq \eta}, \frac{Y_\eta}{\alpha}, \mathcal{N} \right), \quad (33)$$

where \mathcal{N} is a finite random variable independent of Y . Note that for every sequence of real numbers $(r_n)_{n \geq 1}$ such that $r_n \rightarrow \infty$ as $n \rightarrow \infty$ we have $\mathcal{N}_0^{(n)}/r_n \rightarrow 0$ as $n \rightarrow \infty$, in probability.

Set $\beta = (\gamma(1 - \eta))^{1/(2 \wedge \theta)}$. Since $B'_n/B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor} \rightarrow 1/\beta$ as $n \rightarrow \infty$, it immediately follows that:

$$\frac{n - \mathcal{N}_0^{(n)} - j_n(u)/\gamma}{B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor}} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} -\frac{u}{\beta\gamma} \quad (34)$$

We now establish (32). By the definition of $\mathcal{R}_{j_n(u)}^{(n)}$, we have:

$$B'_n \mathcal{R}_{j_n(u)}^{(n)} = \frac{B'_n}{B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor}} \mathbb{E} \left[F(\mathcal{Z}^{(j_n(u))}) B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor} \varphi_{j_n(u) - \lfloor \eta j_n(u) \rfloor} (n - \mathcal{N}_0^{(n)} - \mathcal{Z}_{\lfloor \eta j_n(u) \rfloor}) \right].$$

To simplify notation, set $G_n = B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor} \varphi_{j_n(u) - \lfloor \eta j_n(u) \rfloor} (n - \mathcal{N}_0^{(n)} - \mathcal{Z}_{\lfloor \eta j_n(u) \rfloor})$. By (13), we have:

$$\lim_{n \rightarrow \infty} \left| G_n - p_1 \left(\frac{n - \mathcal{N}_0^{(n)} - \mathcal{Z}_{\lfloor \eta j_n(u) \rfloor} - \frac{1}{\gamma}(j_n(u) - \lfloor \eta j_n(u) \rfloor)}{B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor}} \right) \right| = 0.$$

It follows from (33) and (34) that:

$$F(\mathcal{Z}^{(j_n(u))}) G_n \xrightarrow[n \rightarrow \infty]{(d)} F((Y_t)_{0 \leq t \leq \eta}) p_1 \left(-\frac{u}{\beta\gamma} - \frac{Y_\eta}{\alpha} \right).$$

In addition, by (13), there exists a (deterministic) constant $C > 0$ such that $0 \leq G_n \leq C$ for every $n \geq 1$. Using the fact that $B'_n/B'_{j_n(u) - \lfloor \eta j_n(u) \rfloor} \rightarrow 1/\beta$, the convergence (32) follows from the dominated convergence theorem.

Let us now establish the convergence (31). By Lemma 2.10, the convergence

$$\frac{\mathbb{P}_\mu [\kappa_n = j_n(u)]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \xrightarrow[n \rightarrow \infty]{} \frac{1}{\gamma} \quad (35)$$

holds uniformly in $u \in [-A - 1, A + 1]$. Moreover, it is clear from (13) that $B'_n \mathcal{R}_{j_n(u)}^{(n)}$ is bounded uniformly in $n \geq 1$ and $u \in [-A - 1, A + 1]$. The convergence (31) then follows from an application of the dominated convergence theorem after taking into account (35) and (32).

By Fubini's theorem:

$$\begin{aligned} \frac{1}{\beta\gamma} \int_{-\infty}^{\infty} du \mathbb{E} \left[F((Y_t)_{0 \leq t \leq \eta}) p_1 \left(-\frac{u}{\beta\gamma} - \frac{Y_\eta}{\alpha} \right) \right] &= \frac{1}{\beta\gamma} \mathbb{E} \left[F((Y_t)_{0 \leq t \leq \eta}) \int_{-\infty}^{\infty} du p_1 \left(-\frac{u}{\beta\gamma} - \frac{Y_\eta}{\alpha} \right) \right] \\ &= \mathbb{E} [F((Y_t)_{0 \leq t \leq \eta})] \end{aligned}$$

where we have used the fact that p_1 is a probability distribution in the second equality. Hence we can choose $A > 0$ sufficiently large so that in addition to (28) we also have:

$$\left| \frac{1}{\beta\gamma} \int_{-A}^A du \mathbb{E} \left[F((Y_t)_{0 \leq t \leq \eta}) p_1 \left(-\frac{u}{\beta\gamma} - \frac{Y_\eta}{\alpha} \right) \right] - \mathbb{E} [F((Y_t)_{0 \leq t \leq \eta})] \right| \leq \epsilon.$$

By putting (29), (30) and (31) together, we get that for n sufficiently large:

$$\left| \mathbb{E}_\mu [F(\tilde{Z}^{(\kappa_n)}) \mid \zeta(\tau) = n] - \mathbb{E} [F((Y_t)_{0 \leq t \leq \eta})] \right| \leq 2\epsilon.$$

This establishes (27) and completes the proof. \square

Proof of Corollary 1. By Theorem 1 (i), $\Delta(\mathbf{t}_n)/n$ converges in probability toward γ as $n \rightarrow \infty$ so that $B_{\Delta(\mathbf{t}_n)}/B_n$ converges in probability toward $\gamma^{1/(2 \wedge \theta)}$. In addition, the map $Z \mapsto \sup_{s \in (0,1]} (Z_s - Z_{s-})$ is continuous on $\mathbb{D}([0,1], \mathbb{R})$. It follows from Theorem 3 that:

$$\frac{1}{B_n} \max_{1 \leq i \leq \Delta(\mathbf{t}_n)} \zeta_i(\mathbf{t}_n) \xrightarrow[n \rightarrow \infty]{(d)} \frac{1}{\gamma} \sup_{s \in [0,1]} \Delta Y_s.$$

If $\theta \geq 2$, Y is continuous and the first assumption of Corollary 1 follows. If $\theta < 2$, the result easily follows from the fact that the Lévy measure of Y is $\nu(dx) = \mathbb{1}_{\{x>0\}} dx / (\Gamma(-\theta)x^{1+\theta})$. \square

2.5 Height of large conditioned non-generic trees

We now prove Theorem 4. We keep the notation $u_\star(\tau)$ of the Introduction, as well as the notation $u_\star^{(n)}, \kappa_n, \mathcal{F}_{i,j}^{(n)}$ introduced in the beginning of Section 2.3. Recall also the notation $\mathbb{P}_{\mu,j}$ for the law of a forest of j independent GW_μ trees. If $\mathbf{f} = (\tau_1, \dots, \tau_k)$ is a forest, its height $\mathcal{H}(\mathbf{f})$ is by definition $\max(\mathcal{H}(\tau_1), \dots, \mathcal{H}(\tau_k))$.

Proof of Theorem 4. If τ is a tree, let $\mathcal{H}_\star^{(n)}(\tau) = \mathcal{H}(T_{u_\star^{(n)}} \tau)$ be the height of the subtree of descendants of $u_\star^{(n)}$ in τ . Recall that by Corollary 2.4, $u_\star^{(n)}(\mathbf{t}_n) = u_\star(\mathbf{t}_n)$ with probability tending to one as $n \rightarrow \infty$ and that by Theorem 2 (ii), the generation of $u_\star^{(n)}(\mathbf{t}_n)$ converges in distribution. It is thus sufficient to establish that, if $(\lambda_n)_{n \geq 1}$ of positive real numbers tending to infinity:

$$\mathbb{P} \left[\left| \mathcal{H}_\star^{(n)}(\mathbf{t}_n) - \frac{\ln(n)}{\ln(1/\mathbf{m})} \right| \leq \lambda_n \right] \xrightarrow[n \rightarrow \infty]{} 1. \quad (36)$$

To simplify notation, set $\mathcal{H}_{i,j}^{(n)}(\tau) = \mathcal{H}(\mathcal{F}_{i,j}^{(n)})$ and $\zeta_{i,j}^{(n)} = \zeta(\mathcal{F}_{i,j}^{(n)})$. Set $p_n = \ln(n)/\ln(1/\mathbf{m}) - \lambda_n$. Let us first prove the lower bound, that is $\mathbb{P}[\mathcal{H}_\star^{(n)}(\mathbf{t}_n) \leq p_n] \rightarrow 0$ as $n \rightarrow \infty$. Fix $0 < \epsilon < \gamma$. As in the proof of Theorem 3, we can choose $A > 0$ such that for n sufficiently large:

$$\mathbb{P}_\mu \left[\left| \frac{\kappa_n(\tau) - \gamma n}{B'_n} \right| \geq A \mid \zeta(\tau) = n \right] < \epsilon.$$

Then for n large enough:

$$\begin{aligned}
\mathbb{P} [\mathcal{H}_\star^{(n)}(\mathbf{t}_n) \leq p_n] &= \mathbb{P}_\mu [\mathcal{H}_{1, \lfloor \kappa_n \rfloor}^{(n)} \leq p_n - 1 \mid \zeta(\tau) = n] \\
&\leq \mathbb{P}_\mu [\mathcal{H}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)} \leq p_n \mid \zeta(\tau) = n] \\
&\leq \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu [\mathcal{H}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)} \leq p_n, \kappa_n = j, \zeta(\tau) = n]}{\mathbb{P}_\mu [\zeta(\tau) = n]} + \epsilon \\
&= \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu [\mathcal{H}_{1, \lfloor j/2 \rfloor}^{(n)} \leq p_n, \kappa_n = j, \zeta_{1, \lfloor j/2 \rfloor}^{(n)} + \zeta_{\lfloor j/2 \rfloor + 1, j}^{(n)} + N_0^{(n)} = n]}{\mathbb{P}_\mu [\zeta(\tau) = n]} + \epsilon
\end{aligned}$$

To simplify notation, let $\mathbf{f}_{\lfloor j/2 \rfloor}$ and $\mathcal{N}_0^{(n)}$ be two independent random variables defined under \mathbb{P} such that $\mathbf{f}_{\lfloor j/2 \rfloor}$ is distributed according to $\mathbb{P}_{\mu, \lfloor j/2 \rfloor}$ and $\mathcal{N}_0^{(n)}$ has the same law as $N_0^{(n)}$ under $\mathbb{P}_\mu [\cdot \mid \text{Cut}^{(n)} \neq \partial]$. Then, by Lemma 2.8:

$$\begin{aligned}
&\frac{1}{\mathbb{P}_\mu [\zeta(\tau) = n]} \mathbb{P}_\mu [\mathcal{H}_{1, \lfloor j/2 \rfloor}^{(n)} \leq p_n, \kappa_n = j, \zeta_{1, \lfloor j/2 \rfloor}^{(n)} + \zeta_{\lfloor j/2 \rfloor + 1, j}^{(n)} + N_0^{(n)} = n] \\
&= \frac{\mathbb{P}_\mu [\kappa_n = j]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \mathbb{P}_\mu [\mathcal{H}_{1, \lfloor j/2 \rfloor}^{(n)} \leq p_n, \zeta_{1, \lfloor j/2 \rfloor}^{(n)} + \zeta_{\lfloor j/2 \rfloor + 1, j}^{(n)} + N_0^{(n)} = n \mid \kappa_n = j] \\
&= \frac{\mathbb{P}_\mu [\kappa_n = j]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \mathbb{E} \left[\mathbb{1}_{\{\mathcal{H}(\mathbf{f}_{\lfloor j/2 \rfloor}) \leq p_n\}} \cdot \varphi_{j - \lfloor j/2 \rfloor} (n - \mathcal{N}_0^{(n)} - \zeta(\mathbf{f}_{\lfloor j/2 \rfloor})) \right]
\end{aligned}$$

where we use the notation $\varphi_j(k) = \mathbb{P}_{\mu, j} [\zeta(\mathbf{f}) = k]$ introduced in Section 2.2. It follows that:

$$\mathbb{P} [\mathcal{H}_\star^{(n)}(\mathbf{t}_n) \leq p_n] \leq \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu [\kappa_n = j]}{\mathbb{P}_\mu [\zeta(\tau) = n]} \mathbb{E} \left[\mathbb{1}_{\{\mathcal{H}(\mathbf{f}_{\lfloor j/2 \rfloor}) \leq p_n\}} \cdot \varphi_{j - \lfloor j/2 \rfloor} (n - \mathcal{N}_0^{(n)} - \zeta(\mathbf{f}_{\lfloor j/2 \rfloor})) \right] + \epsilon. \quad (37)$$

We now claim that it is sufficient to establish that for n sufficiently large, for every $j \in [\gamma n - AB'_n, \gamma n + AB'_n]$ and $k \in \mathbb{Z}$ the following three estimates hold with some constants C_1 and C_2 :

$$\begin{cases} \varphi_{j - \lfloor j/2 \rfloor}(k) \leq C_1/B'_n. \end{cases} \quad (38)$$

$$\begin{cases} \mathbb{P}_\mu [\kappa_n = j] \leq C_2 \mathbb{P}_\mu [\zeta(\tau) = n] \end{cases} \quad (39)$$

$$\begin{cases} \mathbb{P} [\mathcal{H}(\mathbf{f}_{\lfloor j/2 \rfloor}) \leq p_n] \leq \epsilon/A \end{cases} \quad (40)$$

Indeed, from (37), we will then get that

$$\mathbb{P} [\mathcal{H}_\star^{(n)}(\mathbf{t}_n) \leq p_n] \leq ((2A + 1)C_1C_2/A + 1)\epsilon,$$

and the lower bound will follow.

The bound (38) is an immediate consequence of the local limit theorem (13). The estimate (39) follows from Lemma 2.10. Let us finally establish (40). Note that for n sufficiently large all the indices j appearing in the sum (37) satisfy $j \geq (\gamma - \epsilon)n$. Consequently, for n sufficiently large:

$$\mathbb{P} [\mathcal{H}(\mathbf{f}_{\lfloor j/2 \rfloor}) \leq p_n] \leq (1 - \mathbb{P}_\mu [\mathcal{H}(\tau) > p_n])^{\lfloor (\gamma - \epsilon)n/2 \rfloor}. \quad (41)$$

Since μ satisfies Assumption (H_θ) , we have $\sum_{i \geq 1} i \ln(i) \mu_i < \infty$. It follows from [14, Theorem 2] that there exists a constant $c > 0$ such that:

$$\mathbb{P}_\mu [\mathcal{H}(\tau) > k] \underset{k \rightarrow \infty}{\sim} c \cdot \mathbf{m}^k. \quad (42)$$

Hence $\mathbb{P}_\mu[\mathcal{H}(\tau) > p_n] \sim c \cdot 1/(n \cdot \mathbf{m}^{\lambda_n})$ as $n \rightarrow \infty$. The right-hand side of (41) consequently tends to 0 as $n \rightarrow \infty$. The estimate (40) follows, and the proof of the lower bound is complete.

Set $q_n = \ln(n)/\ln(1/\mathbf{m}) + \lambda_n$. The proof of the fact that $\mathbb{P}[\mathcal{H}_\star^{(n)}(\mathbf{t}_n) \geq q_n] \rightarrow 0$ as $n \rightarrow \infty$ is similar and we only sketch the argument. Write:

$$\mathbb{P}[\mathcal{H}_\star^{(n)}(\mathbf{t}_n) \geq q_n] \leq \mathbb{P}_\mu[\mathcal{H}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)} \geq q_n \mid \zeta(\tau) = n] + \mathbb{P}_\mu[\mathcal{H}_{\lfloor \kappa_n/2 \rfloor + 1, \kappa_n}^{(n)} \geq q_n \mid \zeta(\tau) = n]$$

Under \mathbb{P}_μ , $\mathcal{H}_{\lfloor \kappa_n/2 \rfloor + 1, \kappa_n}^{(n)}$ has the same distribution as $\mathcal{H}_{1, \kappa_n - \lfloor \kappa_n/2 \rfloor}^{(n)}$. It thus suffices to show that the first term of the last sum tends to 0 as $n \rightarrow \infty$. By similar arguments as in the proof of the lower bound, it is enough to verify that

$$\mathbb{P}_{\mu, \lfloor (\gamma + \epsilon)n/2 \rfloor}[\mathcal{H}(\mathbf{f}) \geq q_n] \xrightarrow{n \rightarrow \infty} 0.$$

This follows from (42), the fact that $\mathbb{P}_{\mu, \lfloor (\gamma + \epsilon)n/2 \rfloor}[\mathcal{H}(\mathbf{f}) \geq q_n] = 1 - (1 - \mathbb{P}_\mu[\mathcal{H}(\tau) \geq q_n])^{\lfloor (\gamma + \epsilon)n/2 \rfloor}$ combined with the asymptotic behavior $\mathbb{P}_\mu[\mathcal{H}(\tau) \geq q_n] \sim c \cdot \mathbf{m}^{\lambda_n}/n$ as $n \rightarrow \infty$. This completes the proof of the upper bound and establishes (36). \square

2.6 Scaling limits of non-generic trees

We turn to the proof of Theorem 5. Recall the notation u_\star , $u_\star^{(n)}$, κ_n , $\mathcal{F}_{i,j}^{(n)}$ and $\varphi_j(k)$ from the Introduction and Section 2.3.

Proof of Theorem 5. Fix $\eta \in (0, 1/\ln(1/\mathbf{m}))$. We shall show that, with probability tending to one as $n \rightarrow \infty$, at least $\ln(n)$ trees among the $\lfloor \kappa_n(\mathbf{t}_n)/2 \rfloor$ trees of $\mathcal{F}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)}(\mathbf{t}_n)$ have height at least $\eta \ln(n)$. This will indeed show that, with probability tending to one as $n \rightarrow \infty$, the number of balls of radius at most η needed to cover $\ln(n)^{-1} \cdot \mathbf{t}_n$ tends to infinity. By standard properties of the Gromov-Hausdorff topology (see [7, Proposition 7.4.12]) this implies that the sequence of random metric spaces $(\ln(n)^{-1} \cdot \mathbf{t}_n)_{n \geq 1}$ is not tight.

If $\mathbf{f} = (\tau_1, \dots, \tau_j)$ is a forest, let $\mathbf{E}(\mathbf{f})$ be the event defined by

$$\mathbf{E}(\mathbf{f}) = \{\text{at most } \ln(n) \text{ trees among } \tau_1, \dots, \tau_j \text{ have height at least } \eta \ln(n)\}.$$

It is thus sufficient to prove that $\mathbb{P}[\mathbf{E}(\mathcal{F}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)}(\mathbf{t}_n))]$ converges toward 0 as $n \rightarrow \infty$. Fix $\epsilon > 0$. As in the proof of Lemma 2.9, we can choose $A > 0$ such that $\mathbb{P}[|\kappa_n(\mathbf{t}_n) - \gamma n| \geq AB'_n] \leq \epsilon$ for n sufficiently large. It is thus sufficient to show that $\mathbb{P}[\mathbf{E}(\mathcal{F}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)}), |\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n] \leq \epsilon$ for n sufficiently large. We keep the notation $\mathbf{f}_{\lfloor j/2 \rfloor}$ and $\mathcal{N}_0^{(n)}$ introduced in the previous section. The same argument that led us to (37) gives

$$\begin{aligned} & \mathbb{P}[\mathbf{E}(\mathcal{F}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)}), |\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n] \\ &= \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}[\mathbf{E}(\mathbf{f}_{\lfloor j/2 \rfloor}) \cdot \varphi_{j - \lfloor j/2 \rfloor}(n - \mathcal{N}_0^{(n)} - \zeta(\mathbf{f}_{\lfloor j/2 \rfloor}))], \end{aligned}$$

Using (38) and (39) we get the existence of a constant $C > 0$ such that for every n sufficiently large

$$\mathbb{P}[\mathbf{E}(\mathcal{F}_{1, \lfloor \kappa_n/2 \rfloor}^{(n)}), |\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n] \leq \frac{C}{B'_n} \sum_{|j - \gamma n| \leq AB'_n} \mathbb{P}[\mathbf{E}(\mathbf{f}_{\lfloor j/2 \rfloor})].$$

It is thus sufficient to prove that for n sufficiently large, if $|j - \gamma n| \leq AB'_n$ then we have $\mathbb{P}[\mathbb{E}(\mathbf{f}_{\lfloor j/2 \rfloor})] \leq \epsilon$. To this end, write

$$\mathbb{P}[\mathbb{E}(\mathbf{f}_{\lfloor j/2 \rfloor})] = \sum_{k=0}^{\ln(n)} \binom{\lfloor j/2 \rfloor}{k} \mathbb{P}_\mu[\mathcal{H}(\tau) > \eta \ln(n)]^k (1 - \mathbb{P}_\mu[\mathcal{H}(\tau) > \eta \ln(n)])^{\lfloor j/2 \rfloor - k}$$

Note that $|j - \gamma n| \leq AB'_n$ implies that $j \leq 2\gamma n$ and $j/2 - \ln(n) \geq \gamma n/3$ for every n sufficiently large. Using (42) and setting $\eta' = \eta \ln(1/m)$, we get that for a certain constant $C > 0$, for every n sufficiently large,

$$\mathbb{P}[\mathbb{E}(\mathbf{f}_{\lfloor j/2 \rfloor})] \leq \sum_{k=0}^{\ln(n)} j^{\ln(n)} \left(1 - \frac{C}{n^{\eta'}}\right)^{j/2-k} \leq \ln(n) \cdot (2\gamma n)^{\ln n} \cdot \exp\left(-\frac{\gamma C}{3} \frac{n}{n^{\eta'}}\right),$$

which tends to 0 as $n \rightarrow \infty$, uniformly in j . This completes the proof. \square

Note that Theorem 5 implies that there is no nontrivial scaling limit for the contour function coding \mathbf{t}_n , since convergence of scaled contour functions imply convergence in the Gromov-Hausdorff topology (see e.g. [27, Lemma 2.3]).

2.7 Finite dimensional marginals of the height function

To prove Theorem 6, we first need to extend the definitions of the height function and Lukasiewicz path to a forest. If $\mathbf{f} = (\tau_i)_{1 \leq i \leq j}$ is a forest, set $n_0 = 0$ and $n_p = \zeta(\tau_1) + \zeta(\tau_2) + \dots + \zeta(\tau_p)$ for $1 \leq p \leq j$. Then, for every $0 \leq i \leq p-1$ and $0 \leq k < \zeta(\tau_{i+1})$, set

$$\mathbf{H}_{n_i+k}(\mathbf{f}) = H_k(\tau_{i+1}), \quad \mathcal{W}_{n_i+k}(\mathbf{f}) = \mathcal{W}_k(\tau_{i+1}) - i.$$

Note that the excursions of $\mathbf{H}(\mathbf{f})$ above 0 are the $(\mathbf{H}_{n_i+k}(\mathbf{f}); 0 \leq k \leq \zeta(\tau_{i+1}))$ and that the excursions of $\mathcal{W}(\mathbf{f})$ above its infimum are the $(\mathcal{W}_{n_i+k}(\mathbf{f}) - i; 0 \leq k \leq \zeta(\tau_{i+1}))$. Finally, for $j \geq 1$, set $\zeta_j = \inf\{n \geq 0; W_n = -j\}$ and for $n \geq 0$, set

$$H_n = \text{Card}(\{k \in \{0, 1, \dots, n-1\}; W_k = \inf_{k \leq j \leq n} W_j\})$$

We will also need the following extension of Proposition 1.4 (see e.g. [27, Proposition 1.7] for a proof):

Proposition 2.11. *Let $j \geq 1$ be an integer. Under $\mathbb{P}_{\mu,j}$, $(\mathbf{H}_i(\mathbf{f}), \mathcal{W}_i(\mathbf{f}))_{0 \leq i \leq \zeta_j(\mathbf{f})}$ has the same distribution as $(H_i, W_i)_{0 \leq i \leq \zeta_j}$.*

In particular, note that $\mathbb{P}_{\mu,j}[\zeta(\mathbf{f}) = n] = \mathbb{P}[\zeta_j = n]$. Finally, the following result, which is an unconditioned version of Theorem 6, will be useful.

Lemma 2.12. *For every $0 < s < t < 1$ and for every $b \geq 0$, the following convergence holds in distribution:*

$$\left(H_{\lfloor ns-b \rfloor}, H_{\lfloor nt-b \rfloor}, \frac{W_{\lfloor nt-b \rfloor} + \gamma nt}{B_{\lfloor nt \rfloor}}\right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_1, \mathbf{e}_2, Y_t)$$

where $\mathbf{e}_1, \mathbf{e}_2$ and Y_t are independent.

Proof. To simplify, we suppose that $b = 0$, the proof being the same in the general case. For every $k \geq 0$, set $M_k = \text{Card}(\{1 \leq i \leq k; W_i = \max_{0 \leq j \leq i} W_j\})$. Set also $W_i^{(n)} = W_{\lfloor nt \rfloor - \lfloor ns \rfloor + i} - W_{\lfloor nt \rfloor - \lfloor ns \rfloor}$ for $i \geq 0$. Notice that $(W_i^{(n)}, i \geq 0)$ has the same distribution as $(W_i, i \geq 0)$. Define similarly $M_k^{(n)} = \text{Card}(\{1 \leq i \leq k; W_i^{(n)} = \max_{0 \leq j \leq i} W_j^{(n)}\})$. Since $(W_i, 0 \leq i \leq n)$ and $(W_n - W_{n-i}, 0 \leq i \leq n)$ have the same distribution, we have

$$\left(H_{\lfloor ns \rfloor}, H_{\lfloor nt \rfloor}, \frac{W_{\lfloor nt \rfloor} + \gamma nt}{B_{\lfloor nt \rfloor}} \right) \stackrel{(d)}{=} \left(M_{\lfloor ns \rfloor}^{(n)}, M_{\lfloor nt \rfloor}^{(n)}, \frac{W_{\lfloor nt \rfloor} + \gamma nt}{B_{\lfloor nt \rfloor}} \right).$$

It is thus sufficient to check that the last expression converges in distribution toward $(\mathbf{e}_1, \mathbf{e}_2, Y_t)$. To this end, first set

$$T = \sup\{i \geq 0, W_i = \sup_{j \geq 0} W_j\}, \quad T^{(n)} = \sup\{i \geq 0, W_i^{(n)} = \sup_{j \geq 0} W_j^{(n)}\},$$

which have the same distribution. Since W drifts almost surely to $-\infty$, T is almost surely finite, and M_T is distributed according to a geometric random variable of parameter $\mathbb{P}[\forall i \geq 1, W_i \leq -1]$. By [33, Theorem 1 in Chapter 2], we have $\mathbb{P}[\forall i \geq 1, W_i \leq -1] = -\mathbb{E}[W_1] = 1 - \mathbf{m}$.

Let $F_1, F_2 : \mathbb{Z} \rightarrow \mathbb{R}_+$ be bounded functions and let $G : \mathbb{R} \rightarrow \mathbb{R}_+$ a bounded uniformly continuous function. Next choose $N_0 > 0$ such that $\mathbb{P}[T > N_0] < \epsilon$. For $i \geq N_0$, note that $M_i^{(n)} = M_{N_0}^{(n)}$ on the event $T^{(n)} \leq N_0$ and that $M_i = M_{N_0}$ on the event $T \leq N_0$. Thus, for $n > N_0/s$,

$$\left| \mathbb{E} \left[F_1(M_{\lfloor ns \rfloor}^{(n)}) F_2(M_{\lfloor nt \rfloor}) G \left(\frac{W_{\lfloor nt \rfloor} + \gamma nt}{B_{\lfloor nt \rfloor}} \right) \right] - \mathbb{E} \left[F_1(M_{N_0}^{(n)}) F_2(M_{N_0}) G \left(\frac{W_{\lfloor nt \rfloor} + \gamma nt}{B_{\lfloor nt \rfloor}} \right) \right] \right| \leq C\epsilon$$

where $C > 0$ is a constant depending only on F_1, F_2, G (and which may change from line to line). For n sufficiently large, using the fact that G is uniformly continuous, one also has

$$\begin{aligned} & \left| \mathbb{E} \left[F_1(M_{N_0}^{(n)}) F_2(M_{N_0}) G \left(\frac{W_{\lfloor nt \rfloor} + \gamma nt}{B_{\lfloor nt \rfloor}} \right) \right] \right. \\ & \quad \left. - \mathbb{E} \left[F_1(M_{N_0}^{(n)}) F_2(M_{N_0}) G \left(\frac{W_{\lfloor nt \rfloor} - W_{N_0} - W_{N_0}^{(n)} + \gamma nt}{B_{\lfloor nt \rfloor}} \right) \right] \right| \leq 2C\epsilon. \end{aligned}$$

Using the fact that $M_{N_0}, M_{N_0}^{(n)}$ and $W_{\lfloor nt \rfloor} - W_{N_0} - W_{N_0}^{(n)}$ are independent for $n > N_0/(s \wedge (t-s))$, we get that, for n sufficiently large,

$$\left| \mathbb{E} \left[F_1(M_{N_0}^{(n)}) F_2(M_{N_0}) G \left(\frac{W_{\lfloor nt \rfloor} - W_{N_0} - W_{N_0}^{(n)} + \gamma nt}{B_{\lfloor nt \rfloor}} \right) \right] - \mathbb{E}[F_1(M_{N_0}^{(n)})] \mathbb{E}[F_2(M_{N_0})] \mathbb{E}[G(Y_t)] \right|$$

is less than $C\epsilon$. Since $|\mathbb{E}[F_1(M_{N_0}^{(n)})] - \mathbb{E}[F_1(M_{T^{(n)}}^{(n)})]| \leq C\epsilon$ and $|\mathbb{E}[F_2(M_{N_0})] - \mathbb{E}[F_2(M_T)]| \leq C\epsilon$, we finally conclude that for n sufficiently large,

$$\left| \mathbb{E} \left[F_1(M_{\lfloor ns \rfloor}^{(n)}) F_2(M_{\lfloor nt \rfloor}) G \left(\frac{W_{\lfloor nt \rfloor} + \gamma nt}{B_{\lfloor nt \rfloor}} \right) \right] - \mathbb{E}[F_1(M_{T^{(n)}}^{(n)})] \mathbb{E}[F_2(M_T)] \mathbb{E}[G(Y_t)] \right| \leq C\epsilon.$$

This completes the proof. \square

Remark 2.13. It is straightforward to adapt the proof of Lemma 2.12 to get that for every $0 < t_1 < t_2 < \dots < t_k < 1$ and for every $b \geq 0$, the following convergence holds in distribution:

$$\left(H_{\lfloor nt_1 - b \rfloor}, H_{\lfloor nt_2 - b \rfloor}, \dots, H_{\lfloor nt_k - b \rfloor}, \frac{W_{\lfloor nt_k - b \rfloor} + \gamma nt_k}{B_{\lfloor nt_k \rfloor}} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k, Y_{t_k})$$

where $(\mathbf{e}_i)_{1 \leq i \leq k}$ and Y_{t_k} are independent.

We are now ready to prove Theorem 6. Recall the notation $u_*(\tau)$ of the Introduction as well as the notation $u_*^{(n)}, \kappa_n, \mathcal{F}_{i,j}^{(n)}, N_0^{(n)}$ and $\varphi_j(k)$ introduced in the beginning of Section 2.3.

Proof of Theorem 6. To simplify, we establish Theorem 6 for $k = 2$, the general case being similar. To this end, we fix $0 < s < t < 1$ and shall show that

$$(H_{\lfloor ns \rfloor}(\mathbf{t}_n), H_{\lfloor nt \rfloor}(\mathbf{t}_n)) \xrightarrow[n \rightarrow \infty]{(d)} (1 + \mathbf{e}_0 + \mathbf{e}_1, 1 + \mathbf{e}_0 + \mathbf{e}_2). \quad (43)$$

Let $F : \mathbb{Z}^2 \rightarrow \mathbb{R}_+$ be a bounded function. To simplify notation, if τ is a tree set $G(\tau) = F(H_{\lfloor \zeta(\tau)s \rfloor}(\tau), H_{\lfloor \zeta(\tau)t \rfloor}(\tau))$. As in the proof of Lemma 2.9, we can choose $A > 0$ such that for every n sufficiently large:

$$\left| \mathbb{E}[G(\mathbf{t}_n)] - \mathbb{E}[G(\mathbf{t}_n) \mathbb{1}_{\{|\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n\}}] \right| \leq \epsilon. \quad (44)$$

It is thus sufficient to show that, for an appropriate choice of A , for every n sufficiently large,

$$\left| \mathbb{E}[G(\mathbf{t}_n) \mathbb{1}_{\{|\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n\}}] - \mathbb{E}[F(1 + \mathbf{e}_0 + \mathbf{e}_1, 1 + \mathbf{e}_0 + \mathbf{e}_2)] \right| \leq \epsilon. \quad (45)$$

To this end, write

$$\mathbb{E}[G(\mathbf{t}_n) \mathbb{1}_{\{|\kappa_n(\mathbf{t}_n) - \gamma n| \leq AB'_n\}}] = \sum_{|j - \gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E}_\mu[G(\tau) \mathbb{1}_{\{\zeta(\tau) = n\}} | \kappa_n = j]$$

Now, for every $n \geq 1$ and $j \geq 1$, let $(\mathcal{S}^{(n)}, \mathcal{N}_g^{(n)}, \mathcal{N}_0^{(n)})$ and \mathbf{f}_j be random variables such that:

- $(\mathcal{S}^{(n)}, \mathcal{N}_g^{(n)}, \mathcal{N}_0^{(n)})_{n \geq 1}$ and $(\mathbf{f}_j)_{j \geq 1}$ are independent,
- \mathbf{f}_j is distributed according to $\mathbb{P}_{\mu,j}$,
- if \mathfrak{T}_n is a tree pointed at vertex v_n that has the same distribution as $\text{Cut}^{(n)}$ under the conditioned probability measure $\mathbb{P}_\mu[\cdot | \text{Cut}^{(n)} \neq \partial]$, then $\mathcal{S}^{(n)}$ is the height of v_n , $\mathcal{N}_g^{(n)}$ is the number of vertices of \mathfrak{T}_n which are less than or equal to v_n in the lexicographical order, and finally $\mathcal{N}_0^{(n)}$ is $\zeta(\mathfrak{T}_n)$.

Then, by Lemma 2.8, we have

$$\begin{aligned} & \mathbb{E}_\mu[G(\tau) \mathbb{1}_{\{\zeta(\tau) = n\}} | \kappa_n = j] \\ &= \mathbb{E} \left[F \left(1 + \mathcal{S}^{(n)} + \mathbf{H}_{\lfloor ns - \mathcal{N}_g^{(n)} \rfloor}(\mathbf{f}_j), 1 + \mathcal{S}^{(n)} + \mathbf{H}_{\lfloor nt - \mathcal{N}_g^{(n)} \rfloor}(\mathbf{f}_j) \right) \mathbb{1}_{\{\mathcal{N}_0^{(n)} + \zeta(\mathbf{f}_j) = n\}} \right]. \end{aligned}$$

By Lemma 2.9, $\mathcal{S}^{(n)}$ converges in distribution toward the height of u_* in $\widehat{\mathcal{T}}$, which is distributed according to a geometric random variable of parameter $1 - \mathbf{m}$. Thus (45) will follow if we manage to check that, for fixed integers $a, b \geq 0$,

$$\left| \sum_{|j-\gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E} [F(\mathbf{H}_{[ns-b]}(\mathbf{f}_j), \mathbf{H}_{[nt-b]}(\mathbf{f}_j)) \mathbb{1}_{\{a+\zeta(\mathbf{f}_j)=n\}}] - \mathbb{E} [F(\mathbf{e}_1, \mathbf{e}_2)] \right| \leq \epsilon \quad (46)$$

for n sufficiently large and for an appropriate choice of A which will be specified later. To this end, using the notation introduced in the beginning of this section and Proposition 2.11, write

$$\mathbb{E} [F(\mathbf{H}_{[ns-b]}(\mathbf{f}_j), \mathbf{H}_{[nt-b]}(\mathbf{f}_j)) \mathbb{1}_{\{a+\zeta(\mathbf{f}_j)=n\}}] = \mathbb{E} [F(H_{[ns-b]}, H_{[nt-b]}) \mathbb{1}_{\{\zeta_j=n-a\}}],$$

Set $\beta = (\gamma(1-t))^{1/(2\wedge\theta)}$, $\beta' = (t/(1-t))^{1/(2\wedge\theta)}$ and $j_n(u) = \lfloor \gamma n + uB'_n \rfloor$ for $u \in \mathbb{R}$. We claim that for fixed $|u| \leq A+1$

$$B'_n \mathbb{E} [F(H_{[ns-b]}, H_{[nt-b]}) \mathbb{1}_{\{\zeta_{j_n(u)}=n-a\}}] \xrightarrow{n \rightarrow \infty} \mathbb{E} [F(\mathbf{e}_1, \mathbf{e}_2)] \cdot \mathbb{E} \left[\frac{1}{\beta} p_1 \left(-\beta' \cdot Y_t - \frac{u}{\gamma\beta} \right) \right] \quad (47)$$

To establish this convergence we use the following Lemma, whose proof is postponed to the end of this section.

Lemma 2.14. *Fix $A > 0$. There exist two constants $c, \delta > 0$ such that for every $n \geq 1$ and $|u| \leq A+1$*

$$(i) \quad \mathbb{P} [W_{[nt-b]} + j_n(u) \leq \gamma n(1-t)/2] \leq ce^{-n^\delta}, \quad (ii) \quad \mathbb{P} \left[\inf_{0 \leq i \leq [nt-b]} W_i \leq -j_n(u) \right] \leq ce^{-n^\delta}.$$

To establish (47), we use the Markov property of the random walk W at time $[nt-b]$ to get

$$\begin{aligned} \mathbb{E} [F(H_{[ns-b]}, H_{[nt-b]}) \mathbb{1}_{\{\zeta_j=n-a\}}] \\ = \mathbb{E} [F(H_{[ns-b]}, H_{[nt-b]}) \mathbb{1}_{\{\zeta_j > [nt-b]\}} \cdot \varphi_{W_{[nt-b]}+j}(n-a-[nt-b])]. \end{aligned}$$

To simplify notation, set $U_n(j) = \mathbb{E} [F(H_{[ns-b]}, H_{[nt-b]}) \varphi_{W_{[nt-b]}+j}(n-a-[nt-b])]$. By Lemma 2.14 (ii), for fixed $|u| \leq A+1$,

$$B'_n \cdot \mathbb{P} [\zeta_{j_n(u)} \leq [nt-b]] \leq B'_n \cdot \mathbb{P} \left[\inf_{0 \leq i \leq [nt-b]} W_i \leq -j_n(u) \right] \xrightarrow{n \rightarrow \infty} 0.$$

Thus (47) will follow if we manage to check that for fixed $|u| \leq A+1$

$$B'_n U_n(j_n(u)) \xrightarrow{n \rightarrow \infty} \mathbb{E} [F(\mathbf{e}_1, \mathbf{e}_2)] \cdot \mathbb{E} \left[\frac{1}{\beta} p_1 \left(-\beta' \cdot Y_t - \frac{u}{\gamma\beta} \right) \right] \quad (48)$$

To this end, write

$$\begin{aligned} B'_n U_n(j_n(u)) \\ = B'_n \mathbb{E} [F(H_{[ns-b]}, H_{[nt-b]}) \varphi_{W_{[nt-b]}+j_n(u)}(n-a-[nt-b]) \mathbb{1}_{\{W_{[nt-b]}+j_n(u) \leq \gamma n(1-t)/2\}}] \\ + B'_n \mathbb{E} [F(H_{[ns-b]}, H_{[nt-b]}) \varphi_{W_{[nt-b]}+j_n(u)}(n-a-[nt-b]) \mathbb{1}_{\{W_{[nt-b]}+j_n(u) > \gamma n(1-t)/2\}}] \end{aligned} \quad (49)$$

By the Skorokhod representation theorem, we may assume that the convergence appearing in Lemma 2.12 holds almost surely. Then (13) and Lemma 2.6 imply that

$$B'_n F(H_{[ns-b]}, H_{[nt-b]}) \varphi_{W_{[nt-b]} + j_n(u)}(n - a - [nt - b]) \mathbb{1}_{\{W_{[nt-b]} + j_n(u) > \gamma n(1-t)/2\}} \xrightarrow{n \rightarrow \infty} F(\mathbf{e}_1, \mathbf{e}_2) \frac{1}{\beta} p_1 \left(-\beta' \cdot Y_t - \frac{u}{\gamma\beta} \right) \quad (50)$$

and that the expression appearing in (50) is bounded uniformly in $n \geq 1$ and $|u| \leq A + 1$. In addition, by Lemma 2.14, the expression appearing in (49) tends to 0 as $n \rightarrow \infty$. The convergence (48) then follows from an application of the dominated convergence theorem, and our claim (47) is proved.

To complete the proof we have to check that (46) holds for n sufficiently large. To this end, using Lemma 2.14 (ii), write

$$\begin{aligned} \sum_{|j-\gamma n| \leq AB'_n} \frac{\mathbb{P}_\mu[\kappa_n = j]}{\mathbb{P}_\mu[\zeta(\tau) = n]} \mathbb{E} \left[F(\mathbf{H}_{[ns-b]}(\mathbf{f}_j), \mathbf{H}_{[nt-b]}(\mathbf{f}_j)) \mathbb{1}_{\{a+\zeta(\mathbf{f}_j)=n\}} \right] \\ = \int_{-A+o(1)}^{A+o(1)} du B'_n \frac{\mathbb{P}_\mu[\kappa_n = j_n(u)]}{\mathbb{P}_\mu[\zeta(\tau) = n]} B'_n U_n(j_n(u)) + o(1). \end{aligned} \quad (51)$$

We claim there exists a constant $C > 0$ such that $B'_n U_n(j_n(u)) \leq C$ for every $n \geq 1$ and $|u| \leq A + 1$. Indeed, using (13) and the fact that $(B'_n)_{n \geq 1}$ can be chosen to be increasing, we have, for a certain constant $c > 0$,

$$U_n(j_n(u)) \leq \|F\|_\infty \mathbb{P} \left[W_{[nt-b]} + j_n(u) \leq \gamma n(1-t)/2 \right] + \|F\|_\infty \frac{c}{B'_{\gamma n(1-t)/2}}.$$

Our claim then follows by Lemma 2.14 (i). Hence we are in position to apply the dominated convergence theorem. By (48) and Lemma 2.10, we get that as $n \rightarrow \infty$, the expression appearing in (51) converges to

$$\mathbb{E} [F(\mathbf{e}_1, \mathbf{e}_2)] \int_{-A}^A du \mathbb{E} \left[\frac{1}{\gamma\beta} p_1 \left(-\beta' \cdot Y_t - \frac{u}{\gamma\beta} \right) \right]$$

As in the end of the proof of Theorem 3, this last expression tends to $\mathbb{E} [F(\mathbf{e}_1, \mathbf{e}_2)]$ as $A \rightarrow \infty$. Consequently, by choosing A sufficiently large, (46) holds for n sufficiently large. This completes the proof of Theorem 6. \square

It remains to prove Lemma 2.14.

Proof of Lemma 2.14. To simplify, we suppose that $b = 0$, the general case being similar. To simplify notation, set $\alpha = (1 + \theta)/2 \wedge 2$ and recall the notation $\bar{W}_n = W_n + \gamma n$ for $n \geq 0$. Note that $\mathbb{E} [\bar{W}_1] = 0$ and that $\mathbb{E} [\bar{W}_1^\alpha] < \infty$. Next, using the inequality $e^{-x} - 1 \leq -x + x^\alpha$ valid for every $x \geq 0$, we get that for every $\lambda > 0$

$$\mathbb{E} [e^{-\lambda \bar{W}_1}] - 1 \leq \lambda^\alpha \mathbb{E} [\bar{W}_1^\alpha].$$

Hence there exists a constant $c_1 > 0$ such that $\ln \mathbb{E} [e^{-\lambda \bar{W}_1}] \leq c_1 \lambda^\alpha$ for every $\lambda > 0$. Next, by using Markov's inequality we have

$$\begin{aligned} \mathbb{P} [W_{[nt]} + j_n(u) \leq \gamma n(1-t)/2] &= \mathbb{P} [e^{-\lambda \bar{W}_{[nt]}} \geq e^{-\lambda(\gamma n(1-t)/2 + \gamma [nt] - j_n(u))}] \\ &\leq e^{\lambda(\gamma n(1-t)/2 + \gamma [nt] - j_n(u))} \exp \left([nt] \ln \mathbb{E} [e^{-\lambda \bar{W}_1}] \right). \end{aligned}$$

But for n sufficiently large and for every $|u| \leq A + 1$ we have $\gamma n(1 - t)/2 + \gamma \lfloor nt \rfloor - j_n(u) \leq -\gamma n(1 - t)/3$. Hence, for n sufficiently large, taking $\lambda = 1/n^\alpha$, we get

$$\mathbb{P} [W_{\lfloor nt \rfloor} + j_n(u) \leq \gamma n(1 - t)/2] \leq e^{-\lambda \gamma n(1 - t)/3 + nt c_1 \lambda^\alpha} \leq e^{t c_1 - \gamma(1 - t)/3 \cdot n^{1 - 1/\alpha}}.$$

This establishes (i) since $\alpha > 1$.

For (ii), we apply Doob's maximal inequality with the submartingale $(e^{\lambda(-\bar{W}_n)}, n \geq 0)$ to get:

$$\begin{aligned} \mathbb{P} \left[\inf_{0 \leq i \leq \lfloor nt \rfloor} W_i \leq -j_n(u) \right] &\leq \mathbb{P} \left[\inf_{0 \leq i \leq \lfloor nt \rfloor} \bar{W}_i \leq \gamma \lfloor nt \rfloor - j_n(u) \right] = \mathbb{P} \left[\sup_{0 \leq i \leq \lfloor nt \rfloor} e^{-\lambda \bar{W}_i} \geq e^{-\lambda(\gamma \lfloor nt \rfloor - j_n(u))} \right] \\ &\leq e^{\lambda(\gamma \lfloor nt \rfloor - j_n(u))} \mathbb{E} [e^{-\lambda \bar{W}_1}]^{\lfloor nt \rfloor} \leq e^{\lambda(\gamma \lfloor nt \rfloor - j_n(u)) + nt c_1 \lambda^\alpha}. \end{aligned}$$

Taking $\lambda = 1/n^\alpha$ and noting that $\gamma \lfloor nt \rfloor - j_n(u) \leq -\gamma n(1 - t)/2$ for every n sufficiently large, the conclusion follows. This completes the proof. \square

3 Extensions and comments

We conclude by proposing possible extensions and stating a few open questions.

Other types of conditioning. Throughout this text, we have only considered the case of Galton-Watson trees conditioned on having a fixed total progeny. It is natural to consider different types of conditioning. For instance, for $n \geq 1$, let \mathbf{t}_n^h be a random tree distributed according to $\mathbb{P}_\mu [\cdot | \mathcal{H}(\tau) \geq n]$. In [18, Section 22], Janson has in particular proved that when μ is critical or subcritical, as $n \rightarrow \infty$, \mathbf{t}_n^h converges locally to a random infinite tree \mathcal{T}^* , which is different from $\hat{\mathcal{T}}$. It would be interesting to know whether the theorems of the present work apply in this case.

Another type of conditioning involving the number of leaves has been introduced in [9, 26, 32]. If τ is a tree, denote by $\lambda(\tau)$ the number of leaves of τ (that is the number of individuals with no child). For $n \geq 1$ such that $\mathbb{P}_\mu [\lambda(\tau) = n] > 0$, let \mathbf{t}_n^l be a random tree distributed according to $\mathbb{P}_\mu [\cdot | \lambda(\tau) = n]$. Do results similar to those we have obtained hold when \mathbf{t}_n is replaced by \mathbf{t}_n^l ? We expect the answer to be positive, since a GW_μ tree with n leaves is very close to a GW_μ with total progeny n/μ_0 (see [26] for details), and we believe that the techniques of the present work can be adapted to solve this problem.

Concentration of $\mathcal{H}(\mathbf{t}_n)$ around $\ln(n)/\ln(1/\mathbf{m})$. By Theorem 4, the sequence of random variable $(\mathcal{H}(\mathbf{t}_n) - \ln(n)/\ln(1/\mathbf{m}))_{n \geq 1}$ is tight. It is therefore natural to ask the following question, due to Nicolas Broutin. Does there exist a random variable \mathcal{H} such that:

$$\mathcal{H}(\mathbf{t}_n) - \frac{\ln(n)}{\ln(1/\mathbf{m})} \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{H} \quad ?$$

We expect the answer to be negative. Let us give a heuristic argument to support this prediction. In the proof of Theorem 4, we have seen that the height of $\mathcal{H}(\mathbf{t}_n)$ is close to the height of $\lfloor \gamma n \rfloor$ independent GW_μ trees and the height of each of these trees satisfies the estimate (42). However, if $(Q_i)_{i \geq 1}$ is an i.i.d. sequence of random variables such that $\mathbb{P}[Q_1 \geq k] = c \cdot \mathbf{m}^k$, then it is known (see e.g. [17, Example 4.3]) that the random variables

$$\max(Q_1, Q_2, \dots, Q_n) - \frac{\ln(n)}{\ln(1/\mathbf{m})}$$

do not converge in distribution.

Behavior of $\mathbb{E}[\mathcal{H}(\mathbf{t}_n)]$. We conjecture that:

$$\mathbb{E}[\mathcal{H}(\mathbf{t}_n)] \underset{n \rightarrow \infty}{\sim} \frac{\ln(n)}{\ln(1/\mathbf{m})}.$$

However, it seems that more precise estimates than the ones we have used are needed to prove this statement.

Other types of trees. Janson [18] gives a very general limit theorem concerning the local asymptotic behavior of simply generated trees conditioned on having a fixed large number of vertices. Let us briefly recall the definition of simply generated trees. Fix a sequence $\mathbf{w} = (w_k)_{k \geq 0}$ of nonnegative real numbers such that $w_0 > 0$ and such that there exists $k > 1$ with $w_k > 0$ (\mathbf{w} is called a weight sequence). Let $\mathbb{T}_f \subset \mathbb{T}$ be the set of all finite plane trees and, for every $n \geq 1$, let \mathbb{T}_n be the set of all plane trees with n vertices. For every $\tau \in \mathbb{T}_f$, define the weight $w(\tau)$ of τ by:

$$w(\tau) = \prod_{u \in \tau} w_{k_u(\tau)}.$$

Then for $n \geq 1$ set

$$Z_n = \sum_{\tau \in \mathbb{T}_n} w(\tau).$$

For every $n \geq 1$ such that $Z_n \neq 0$, let \mathcal{T}_n be a random tree taking values in \mathbb{T}_n such that for every $\tau \in \mathbb{T}_n$:

$$\mathbb{P}[\mathcal{T}_n = \tau] = \frac{w(\tau)}{Z_n}.$$

The random tree \mathcal{T}_n is said to be finitely generated. Galton-Watson trees conditioned on their total progeny are particular instances of simply generated trees. Conversely, if \mathcal{T}_n is as above, there exists an offspring distribution μ such that \mathcal{T}_n has the same distribution as a GW_μ tree conditioned on having n vertices if, and only if, the radius of convergence of $\sum w_i z^i$ is positive (see [18, Section 8]).

It would thus be interesting to find out if the theorems obtained in the present work for Galton-Watson trees can be extended to the setting of simply generated trees whose associated radius of convergence is 0. In the latter case, Janson [18] proved that \mathcal{T}_n converges locally as $n \rightarrow \infty$ toward a deterministic tree consisting of a root vertex with an infinite number of leaves attached to it. We thus expect that the asymptotic properties derived in the present work will take a different form in this case. We hope to investigate this in future work.

References

- [1] D. ALDOUS, *The continuum random tree III*, Ann. Probab., 21 (1993), pp. 248–289.
- [2] I. ARMENDÁRIZ AND M. LOULAKIS, *Conditional distribution of heavy tailed random variables on large deviations of their sum*, Stochastic Process. Appl., 121 (2011), pp. 1138–1147.
- [3] K. B. ATHREYA AND P. E. NEY, *Branching processes*, vol. 196 of Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, 1972.

- [4] P. BIALAS, Z. BURDA, AND D. JOHNSTON, *Condensation in the backgammon model*, Nuclear Physics B, 493 (1997), p. 505.
- [5] P. BILLINGSLEY, *Convergence of probability measures*, Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, second ed., 1999. A Wiley-Interscience Publication.
- [6] N. H. BINGHAM, C. M. GOLDIE, AND J. L. TEUGELS, *Regular variation*, vol. 27 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1989.
- [7] D. BURAGO, Y. BURAGO, AND S. IVANOV, *A course in metric geometry*, vol. 33 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.
- [8] N. CURIEN AND I. KORTCHEMSKI, *Percolation on random triangulations and stable loop-trees*, In preparation.
- [9] —, *Random non-crossing plane configurations: a conditioned Galton-Watson tree approach*, Random Structures Algorithms (to appear).
- [10] D. DENISOV, A. B. DIEKER, AND V. SHNEER, *Large deviations for random walks under subexponentiality: the big-jump domain*, Ann. Probab., 36 (2008), pp. 1946–1991.
- [11] T. DUQUESNE, *A limit theorem for the contour process of conditioned Galton-Watson trees*, Ann. Probab., 31 (2003), pp. 996–1027.
- [12] R. DURRETT, *Conditioned limit theorems for random walks with negative drift*, Z. Wahrsch. Verw. Gebiete, 52 (1980), pp. 277–287.
- [13] S. GROSSKINSKY, G. M. SCHÜTZ, AND H. SPOHN, *Condensation in the zero range process: stationary and dynamical properties*, J. Statist. Phys., 113 (2003), pp. 389–410.
- [14] C. R. HEATHCOTE, E. SENETA, AND D. VERE-JONES, *A refinement of two theorems in the theory of branching processes*, Teor. Verojatnost. i Primenen., 12 (1967), pp. 341–346.
- [15] I. A. IBRAGIMOV AND Y. V. LINNIK, *Independent and stationary sequences of random variables*, Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman.
- [16] J. JACOD AND A. N. SHIRYAEV, *Limit theorems for stochastic processes*, vol. 288 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, second ed., 2003.
- [17] S. JANSON, *Rounding of continuous random variables and oscillatory asymptotics*, Ann. Probab., 34 (2006), pp. 1807–1826.
- [18] —, *Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation*, Probab. Surv., 9 (2012), pp. 103–252.

- [19] S. JANSON AND S. O. STEFÁNSSON, *Scaling limits of random planar maps with a unique large face*, arXiv:1212.5072.
- [20] I. JEON, P. MARCH, AND B. PITTEL, *Size of the largest cluster under zero-range invariant measures*, Ann. Probab., 28 (2000), pp. 1162–1194.
- [21] T. JONSSON AND S. O. STEFÁNSSON, *Condensation in nongeneric trees*, J. Stat. Phys., 142 (2011), pp. 277–313.
- [22] D. P. KENNEDY, *The Galton-Watson process conditioned on the total progeny*, J. Appl. Probability, 12 (1975), pp. 800–806.
- [23] H. KESTEN, *Subdiffusive behavior of random walk on a random cluster*, Ann. Inst. H. Poincaré Probab. Statist., 22 (1986), pp. 425–487.
- [24] H. KESTEN AND B. PITTEL, *A local limit theorem for the number of nodes, the height, and the number of final leaves in a critical branching process tree*, Random Structures Algorithms, 8 (1996), pp. 243–299.
- [25] I. KORTCHEMSKI, *A simple proof of Duquesne’s theorem on contour processes of conditioned Galton-Watson trees*, To appear in Séminaire de Probabilités.
- [26] —, *Invariance principles for Galton-Watson trees conditioned on the number of leaves*, Stochastic Process. Appl., 122 (2012), pp. 3126–3172.
- [27] J.-F. LE GALL, *Random trees and applications*, Probability Surveys, (2005).
- [28] —, *Random real trees*, Ann. Fac. Sci. Toulouse Math. (6), 15 (2006), pp. 35–62.
- [29] —, *Itô’s excursion theory and random trees*, Stochastic Process. Appl., 120 (2010), pp. 721–749.
- [30] J. PITMAN, *Combinatorial stochastic processes*, vol. 1875 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2006. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard.
- [31] R. PYKE AND D. ROOT, *On convergence in r -mean of normalized partial sums*, Ann. Math. Statist., 39 (1968), pp. 379–381.
- [32] D. RIZZOLO, *Scaling limits of Markov branching trees and Galton-Watson trees conditioned on the number of vertices with out-degree in a given set*, (2011).
- [33] L. TAKÁCS, *Combinatorial methods in the theory of stochastic processes*, Robert E. Krieger Publishing Co., Huntington, N. Y., 1977. Reprint of the 1967 original.
- [34] V. M. ZOLOTAREV, *One-dimensional stable distributions*, vol. 65 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1986. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.

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